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Information and Entropy in Quantum Measurement Processes

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Quantum measurement processes of discrete and continuous observables are considered from the information-theoretic point of view. The information extracted from the results of quantum measurement performed on a physical system and the change of the Shannon entropy of the measured physical system are investigated in detail. It is shown that the amount of information about the intrinsic observable of the measured physical system can be expressed by the mutual information between the physical system and the measurement apparatus if the intrinsic observable commutes with the operational observable defined by the quantum measurement process. Furthermore, the condition can be obtained under which the amount of information extracted from the measurement outcomes becomes equal to the decrease of the entropy of the measured physical system. In addition, the change of the Shannon entropy is compared with that of the von Neumann entropy. The general results do not depend on whether the readout of the measurement outcome obeys the projection postulate or not. Several examples of quantum measurement processes are considered to examine the general results.

1. INTRODUCTION

The entropy of a physical system is one of the most important quantities in thermodynamics and statistical mechanics (Mayer and Mayer, 1977). For example, the second law of thermodynamics is formulated in terms of entropy. In thermal equilibrium, the entropy of a macroscopic system is obtained by the Boltzmann formula. Let W be the number of microscopic states of the system that are macroscopically equivalent in thermal equilibrium. Then the Boltzmann formula tells us that the entropy H of the system is given by $H = \ln W$, where we set the Boltzmann constant $k_B = 1$. To rewrite the Boltzmann formula, let p_j be the probability that the jth microscopic state

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appears in thermal equilibrium. The principle of equal *a priori* probabilities says that all the probabilities are equal, that is, $p_j = 1/W$ for all j (Mayer and Mayer, 1977). Hence the entropy in thermal equilibrium can be expressed in the form $H = -\Sigma_{j=1}^{W} p_j \ln p_j$. Conversely, when we define the entropy by this formula, the thermal equilibrium state is obtained by applying the entropy-maximum principle (Jaynes, 1957a, b).

The relation between entropy and information was considered first by Szilard (1929), who showed that the information gain by the measurement of the thermodynamic system decreases the entropy of the measured system. Of course, the total entropy of the measured system and the measurement apparatus increases, which is the second law of thermodynamics. The importance of Szilard's work was pointed out by Brillouin (1956), while Szilard's work was criticized by Jauch and Báron (1972). The most important and interesting work to understand the relation between entropy and information was done by Shannon (1948a, b; Shannon and Weaver, 1949) who introduced the entropy, called the Shannon entropy, into communications theory. He showed the source-coding theorem and the channel-coding theorem: The former says that the average length of a code word representing a symbol generated from a message source is lower bounded by the Shannon entropy of the message source; the latter ensures that the information can be reliably transmitted through a noisy channel if the information rate is less than the channel capacity, which is the maximum value of the mutual information of the communication channel. The work by Szilard (1929), Brillouin (1956), and Shannon (1948a, b; Shannon and Weaver, 1949) treated classical systems. Other interesting work on the relations among entropy, information, and randomness of physical systems includes that by Zurek (1989) and Caves (1993).

Quantum mechanical entropy was introduced by von Neumann in the quantum theory of measurement (von Neumann, 1955). Let $\hat{\rho}$ be a statistical operator which describes the quantum state of a physical system. Then the quantum mechanical entropy $S(\hat{\rho})$, called the von Neumann entropy, is given by $S(\hat{\rho}) = -\mathrm{Tr}[\hat{\rho} \ln \hat{\rho}]$, where Tr stands for the trace operation over the Hilbert space on which the statistical operator $\hat{\rho}$ is defined. When a physical system is prepared in the quantum state $\hat{\rho}$, an observable $\hat{A}$ of the physical system, which has the eigenstate $|\psi(a)\rangle$ with eigenvalue a, takes the value a with probability $p_A(a) = \langle\psi(a)|\hat{\rho}|\psi(a)\rangle$. In this case, the Shannon entropy of the observable $\hat{A}$ is given by $H(p_A) = -\Sigma_a p(a) \ln p(a)$, which is no less than the von Neumann entropy $S(\hat{\rho})$, namely, $S(\hat{\rho}) \leq H(p_A)$, where the entropy $H(p_A)$ is referred to as the measurement entropy in some cases (Ballan *et al.*, 1986). The properties of the von Neumann entropy in quantum measurement processes have been investigated by several authors (Groenewold, 1971; Lindblad, 1973; Ozawa, 1986).

It has recently been found that the von Neumann entropy in quantum information theory (Belavkin *et al.*, 1995; Hirota *et al.*, 1997) plays the same role as the Shannon entropy does in classical information theory (Cover and Thomas, 1991). In fact, the two quantum coding theorems have been proven in terms of the von Neumann entropy (Schumacher, 1995; Jozsa and Schumacher, 1994; Schumacher and Westmoreland, 1997; Holevo, 1998). The quantum source-coding theorem says that the average number of quantum bits (qubits) representing a pure quantum state generated from a quantum message source is lower bounded by the von Neumann entropy of the source (Schumacher, 1995; Jozsa and Schumacher, 1994), and the quantum channel-coding theorem ensures that the information can be reliably transmitted through a noisy quantum channel if the information rate is less than the channel capacity, called the Holevo bound (Holevo, 1973), which is calculated in terms of the von Neumann entropy (Schumacher and Westmoreland, 1997; Holevo, 1998). Quantum information theory, which includes quantum computing, quantum coding, and quantum cryptography, is one of the most important subjects in present-day quantum physics and information science (Belavkin *et al.*, 1995; Hirota *et al.*, 1997).

When quantum measurement is performed on a physical system, the quantum state of the measured system inevitably changes due to the effects of the quantum measurement process. Any quantum measurement process that does not disturb the quantum state gives us no information about the measured system. The state change of the measured system induces a change of the Shannon entropy (or the measurement entropy) of the system. Thus it is clear that any quantum measurement process in which the entropy of the system remains unchanged does not give us any information about the system. Therefore it is important to investigate the relation between the amount of information about the physical system extracted from the measurement outcomes and the entropy change of the measured physical system. This is the main subject of this paper. In particular, we would like to obtain the condition for quantum measurement processes under which the amount of information extracted from the measurement outcomes is equal to the decrease of the entropy of the measured physical system. Furthermore, we consider several models of quantum measurement processes to examine the general results.

This paper is organized as follows. In Section 2 we briefly review the basic elements of the quantum theory of measurement in a suitable way for our purpose, and then we introduce probability distributions in a quantum measurement process. In Section 3 we consider the information about the measured physical system that is extracted from measurement outcomes, and we find the condition under which the amount of information about the physical system can be expressed by the mutual information between the physical system and the measurement apparatus. The condition is equivalent to the commutativity of the intrinsic observable of the measured system and the operational observable defined by the quantum measurement process. In Section 4 we investigate the entropy change of the physical system caused by the quantum measurement process. We obtain the condition under which the amount of information extracted from the measurement outcomes becomes equal to the entropy decrease of the measured physical system. Furthermore, we compare the change of the Shannon entropy with that of the von Neumann entropy in the quantum measurement process. In Section 5 we consider the several examples of quantum measurement processes to examine the general results obtained in Sections 3 and 4. We investigate the normal unitary process and the SU(2) and SU(1, 1) processes in quantum optical systems. In Section 6 we consider the entropy change and the information gain in quantum measurement processes of continuous observables. We can obtain the same results as those for quantum measurement processes of discrete observables. As examples, we investigate position and momentum measurements of a physical system in one-dimensional space. In Section 7 we consider continuous quantum measurements, such as the photon counting measurement, to obtain information about a physical system. As an example, we investigate the degenerate four-wave mixing process with the photon counting measurement, which is equivalent to the continuous quantum nondemolition measurement of the photon number of a physical system. In Section 8 we summarize the results.

2. QUANTUM MEASUREMENT PROCESSES

In this section we briefly summarize the basic formulation of quantum measurement processes in a suitable way for our purpose (Busch *et al.*, 1991, 1995; Kraus, 1983), and then we introduce probability distributions that an observable takes some values in the premeasurement and postmeasurement quantum states, by means of which the Shannon entropies are calculated (Shannon, 1948a, b; Shannon and Weaver, 1949). Suppose that we perform quantum measurement on a physical system $\mathcal{S}$ to obtain some information about an observable $\hat{\chi}_S$ in a quantum state which is described by a statistical operator $\hat{\rho}_{\text{in}}^S$ which satisfies $\hat{\rho}_{\text{in}}^S \geq 0$ and $\text{Tr}_S\hat{\rho}_{\text{in}}^S = 1$, where Tr_S stands for the trace operation over the Hilbert space $\mathcal{H}_S$ of the physical system. We assume that the measured observable $\hat{\chi}_S$ of the physical system has a spectral decomposition given by

$$\hat{\chi}_S = \sum_{\mu \in \mathcal{M}} \mu |\psi_S(\mu)\rangle\langle\psi_S(\mu)| = \sum_{\mu \in \mathcal{M}} \mu \hat{E}_\chi^S(\mu) \tag{2.1}$$

where $\mathcal{M}$ represents the spectral set of the observable $\hat{\chi}_S$ and $\hat{E}_\chi^S(\mu)$ is a projection operator onto the one-dimensional eigenspace of $\hat{\chi}_S$. In equation (2.1) we have assumed that the observable $\hat{\chi}_S$ has nondegenerate and discrete eigenvalues, for the sake of simplicity. The set of the eigenstates $S_\chi = \{|\psi_S(\mu)\rangle | \mu \in \mathcal{M}\}$ becomes a complete orthonormal system of the Hilbert space $\mathcal{H}_S$, which satisfies the relations

$$\langle\psi_S(\mu_1)|\psi_S(\mu_2)\rangle = \delta_{\mu_1,\mu_2}, \qquad \sum_{\mu\in\mathcal{M}} |\psi_S(\mu)\rangle\langle\psi_S(\mu)| = \hat{I}_S \tag{2.2}$$

where $\hat{I}_S$ stands for an identity operator defined on the Hilbert space $\mathcal{H}_S$.

To measure the observable $\hat{\chi}_S$, we first have to prepare a measurement apparatus $\mathcal{A}$, the initial quantum state of which is described by a statistical operator $\hat{\rho}^A_{\text{in}}$. We denote the Hilbert space of the measurement apparatus as $\mathcal{H}_A$. We next have the measurement apparatus interact with the physical system to make some quantum correlation between them, which is indispensable for obtaining information about the physical system by means of the measurement apparatus. Let $\mathcal{U}_{SA}$ be a unitary operator that describes the state change of the physical system and the measurement apparatus caused by the interaction. If the interaction is represented by a Hamiltonian $\hat{H}^{SA}_{\text{int}}$ and the interaction time is τ, the unitary operator $\mathcal{U}_{SA}$ is given by

$$\mathcal{U}_{SA} = \exp\left(-\frac{i}{\hbar}\,\tau\,\hat{H}^{SA}_{\text{int}}\right) \tag{2.3}$$

After the interaction, the compound quantum state of the physical system and measurement apparatus becomes

$$\hat{\rho}^{SA}_{\text{out}} = \mathcal{U}_{SA}(\hat{\rho}^S_{\text{in}} \otimes \hat{\rho}^A_{\text{in}})\mathcal{U}^\dagger_{SA} \tag{2.4}$$

In this paper we ignore the individual time evolutions of the physical system and the measurement apparatus, for the sake of simplicity, since they do not affect our results.

We finally perform the readout of the result of the quantum measurement process. The readout of the measurement outcome is mathematically described by a positive operator-valued measure (POVM) defined on the Hilbert space $\mathcal{H}_A$ (Davies, 1976; Helstrom, 1976; Holevo, 1982). The readout which gives the output value ν is described by the POVM $\hat{E}^A_{\mathcal{Y}}(\nu)$, which satisfies the relations

$$\hat{E}^A_{\mathcal{Y}}(\nu) \geq 0, \qquad \sum_{\nu\in\mathcal{N}} \hat{E}^A_{\mathcal{Y}}(\nu) = \hat{I}_A \tag{2.5}$$

where $\hat{I}_A$ is an identity operator defined on the Hilbert space $\mathcal{H}_A$, and $\mathcal{N}$ represents the set of all possible outcomes of the quantum measurement process. If the measurement outcome ν corresponds to the value of some observable $\mathcal{Y}_A$ of the measurement apparatus, the POVM $\hat{E}^A_{\mathcal{Y}}(\nu)$ becomes a projection operator $\hat{E}^A_{\mathcal{Y}}(\nu) = |\phi_A(\nu)\rangle\langle\phi_A(\nu)|$, where $|\phi_A(\nu)\rangle$ is the eigenstate of the observable $\mathcal{Y}_A = \Sigma_{\nu\in\mathcal{N}}\nu\ \hat{E}^A_{\mathcal{Y}}(\nu)$, which is called the pointer observable. The set $\{|\phi_A(\nu)\rangle|\nu \in \mathcal{N}\}$ is referred to as the pointer base. It should be noted that even if the readout of the measurement outcome cannot be performed by measuring any pointer observable $\mathcal{Y}_A$, the results obtained in this paper are still valid since we use only relations (2.2) and (2.5) to derive the results. The photon counting measurement, which is a continuous quantum measurement of photon number (Srinivas and Davies, 1981; Srinivas, 1996; Chmara, 1987), is a typical example that there is not a pointer observable (see Section 7).

When the measurement outcome ν of the quantum measurement process is given, the quantum state of the physical system after the measurement is obtained from (2.4) by means of the state-reduction formula (Kraus, 1983; Ozawa, 1983, 1984)

$$\hat{\rho}^S_{\text{out}}(\nu) = \frac{\text{Tr}_A[(\hat{I}_S \otimes \hat{E}^A_{\mathcal{Y}}(\nu))\hat{\rho}^{SA}_{\text{out}}]}{\text{Tr}_{SA}[(\hat{I}_S \otimes \hat{E}^A_{\mathcal{Y}}(\nu))\hat{\rho}^{SA}_{\text{out}}]} \tag{2.6}$$

where Tr_A and Tr_{SA} stand for the trace operations over the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_{SA} = \mathcal{H}_S \otimes \mathcal{H}_A$. We refer to the quantum state $\hat{\rho}^S_{\text{out}}(\nu)$ as the postmeasurement state of the physical system. The probability $P^A_{\text{out}}(\nu)$ that we obtain the measurement outcome ν is calculated by

$$P^A_{\text{out}}(\nu) = \text{Tr}_{SA}[(\hat{I}_S \otimes \hat{E}^A_{\mathcal{Y}}(\nu))\hat{\rho}^{SA}_{\text{out}}] \tag{2.7}$$

where equation (2.5) ensures that $P^A_{\text{out}}(\nu)$ is nonnegative and normalized as $\Sigma_{\nu\in\mathcal{N}}\ P^A_{\text{out}}(\nu) = 1$. In the postmeasurement state $\hat{\rho}^S_{\text{out}}(\nu)$ of the physical system, the observable $\hat{\chi}_S$ takes the value μ with probability

$$P^S_{\text{out}}(\mu|\nu) = \text{Tr}_S[\hat{E}^S_\chi(\mu)\hat{\rho}^S_{\text{out}}(\nu)] \tag{2.8}$$

which is conditioned by the measurement outcome ν. When we do not perform the readout of the measurement outcome ν, the postmeasurement state $\hat{\rho}^S_{\text{out}}$ of the physical system becomes

$$\hat{\rho}^S_{\text{out}} = \text{Tr}_A\hat{\rho}^{SA}_{\text{out}} \tag{2.9}$$

in which the observable $\hat{\chi}_S$ takes the value μ with probability

$$P^S_{\text{out}}(\mu) = \text{Tr}_S[E^S_\chi(\mu)\hat{\rho}^S_{\text{out}}] = \text{Tr}_{SA}[(\hat{E}^S_\chi(\mu) \otimes \hat{I}_A)\hat{\rho}^{SA}_{\text{out}}] \tag{2.10}$$

The quantum measurement in which we do (do not) perform the readout of the measurement outcome ν is called the selective (nonselective) quantum measurement.

There are several relations among the probabilities. We first obtain from equations (2.7), (2.8), and (2.10)

$$P^S_{\text{out}}(\mu) = \sum_{\nu \in \mathcal{N}} P^S_{\text{out}}(\mu|\nu) P^A_{\text{out}}(\nu) \tag{2.11}$$

Note that we can define the joint probability in the compound quantum state $\hat{\rho}^{SA}_{\text{out}}$ of the physical system and the measurement apparatus,

$$P^{SA}_{\text{out}}(\mu, \nu) = \text{Tr}_{SA}[(\hat{E}^S_{\chi}(\mu) \otimes \hat{E}^A_{\mathcal{Y}}(\nu))\hat{\rho}^{SA}_{\text{out}}] \tag{2.12}$$

Then we obtain the relations among the probabilities

$$P^{SA}_{\text{out}}(\mu, \nu) = P^S_{\text{out}}(\mu|\nu)\ P^A_{\text{out}}(\nu) \tag{2.13}$$

$$P^S_{\text{out}}(\mu) = \sum_{\nu \in \mathcal{N}} P^{SA}_{\text{out}}(\mu, \nu), \qquad P^A_{\text{out}}(\nu) = \sum_{\mu \in \mathcal{M}} P^{SA}_{\text{out}}(\mu, \nu) \tag{2.14}$$

According to the Bayes theorem (Caves and Drummond, 1994), we obtain the *posterior* probability $P^A_{\text{out}}(\nu|\mu)$,

$$P^A_{\text{out}}(\nu|\mu) = \frac{P^{SA}_{\text{out}}(\mu, \nu)}{P^S_{\text{out}}(\mu)} = \frac{P^S_{\text{out}}(\mu|\nu)\ P^A_{\text{out}}(\nu)}{P^S_{\text{out}}(\mu)} \tag{2.15}$$

Using these probabilities, we can calculate the Shannon entropies in the quantum measurement process.

The quantum measurement process performed on a physical system is completely determined by the triplet $\mathbb{M} = \langle \hat{\rho}^A_{\text{in}}, \hat{E}^A_{\mathcal{Y}}(\nu), \mathcal{U}_{SA} \rangle$. Thus far we have not restricted the initial quantum state $\hat{\rho}^A_{\text{in}}$ of the measurement apparatus, the readout process described by the POVM $\hat{E}^A_{\mathcal{Y}}(\nu)$, or the interaction between the physical system and the measurement apparatus given by the unitary operator $\mathcal{U}_{SA}$. The triplet $\mathbb{M} = \langle \hat{\rho}^A_{\text{in}}, \hat{E}^A_{\mathcal{Y}}(\nu), \mathcal{U}_{SA} \rangle$ is usually determined so as to satisfy *the probability reproducibility condition* (Busch *et al.*, 1991, 1995) given by the relation

$$P^S_{\text{in}}(\mu) = P^A_{\text{out}}(\nu), \qquad \mu = f(\nu) \tag{2.16}$$

where $P^S_{\text{in}}(\mu)$ represents the probability that the observable $\hat{\chi}_S$ takes the value μ in the premeasurement state $\hat{\rho}^S_{\text{in}}$ of the physical system,

$$P^S_{\text{in}}(\mu) = \text{Tr}_S[\hat{E}^S_{\chi}(\mu)\hat{\rho}^S_{\text{in}}] \tag{2.17}$$

and $f(\nu)$ is some analytic function which connects the measurement outcome ν with the value μ taken by the observable $\hat{\chi}_S$ of the physical system. In this paper, however, we do not impose the probability reproducibility condition on quantum measurement processes and we will consider the information and the entropy change in a quantum measurement process characterized by an arbitrary triplet $\mathbb{M} = \langle \hat{\rho}^A_{\text{in}}, \hat{E}^A_{\mathcal{Y}}(\nu), \mathcal{U}_{SA} \rangle$.

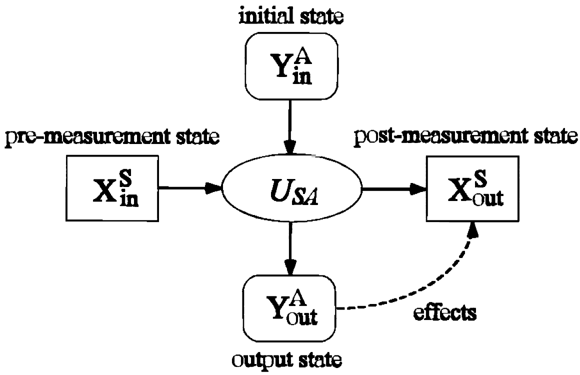

Fig. 1. Schematic representation of the quantum measurement process.

Before closing this section, we summarize the notation used throughout this paper. In the premeasurement state $\hat{\rho}^S_{in}$ of the physical system, the observable $\hat{\chi}_S$ takes the value μ with the probability $P^S_{in}(\mu)$. This defines the random variable X^S_{in} in the premeasurement state of the physical system. In the same way we can define the random variable X^S_{out} in the postmeasurement state of the physical system. Since the measurement outcome ν of the quantum measurement process is governed by the probability $P^A_{out}(\nu)$, we can define the random variable Y^A_{out} in the output state of the measurement apparatus. We can also introduce the random variable Y^A_{in} in the initial state of the measurement apparatus. The quantum measurement process that we consider is schematically shown in Fig. 1 in terms of these random variables. Using the probabilities introduced above, we can calculate the Shannon entropies (or the measurement entropies) (Cover and Thomas, 1991) in the quantum measurement process,

$$H(X^S_{in}) = -\sum_{\mu\in\mathcal{M}} P^S_{in}(\mu) \ln P^S_{in}(\mu) \tag{2.18}$$

$$H(X^S_{out}) = -\sum_{\mu\in\mathcal{M}} P^S_{out}(\mu) \ln P^S_{out}(\mu) \tag{2.19}$$

$$H(Y^A_{out}) = -\sum_{\nu\in\mathcal{N}} P^A_{out}(\nu) \ln P^A_{out}(\nu) \tag{2.20}$$

$$H(X^S_{out}, Y^A_{out}) = -\sum_{\mu\in\mathcal{M}} \sum_{\nu\in\mathcal{N}} P^{SA}_{out}(\mu, \nu) \ln P^{SA}_{out}(\mu, \nu) \tag{2.21}$$

Furthermore, the conditional entropies are given by

$$H(X^S_{out}|Y^A_{out}) = -\sum_{\mu\in\mathcal{M}} \sum_{\nu\in\mathcal{N}} P^{SA}_{out}(\mu, \nu) \ln P^S_{out}(\mu|\nu) \tag{2.22}$$

$$H(Y^A_{out}|X^S_{out}) = -\sum_{\mu\in\mathcal{M}} \sum_{\nu\in\mathcal{N}} P^{SA}_{out}(\mu, \nu) \ln P^A_{out}(\nu|\mu) \tag{2.23}$$

The well-known relations among the entropies are obtained from (2.15),

$$H(X^S_{out}, Y^A_{out}) = H(X^S_{out}|Y^A_{out}) + H(Y^A_{out})$$
$$= H(Y^A_{out}|X^S_{out}) + H(X^S_{out}) \tag{2.24}$$

The mutual information $H(X^S_{out}; Y^A_{out})$ is calculated by

$$H(X^S_{out}; Y^A_{out}) = H(X^S_{out}) - H(X^S_{out}|Y^A_{out})$$
$$= H(X^S_{out}) + H(Y^A_{out}) - H(X^S_{out}, Y^A_{out})$$
$$= H(Y^A_{out}) - H(Y^A_{out}|X^S_{out}) \tag{2.25}$$

Note that the Shannon mutual information is symmetric with respect to the random variables, that is, $H(X^S_{out}; Y^A_{out}) = H(Y^A_{out}; X^S_{out})$. These entropies are used to investigate the entropy change of the physical system in the quantum measurement process.

3. INFORMATION GAIN IN QUANTUM MEASUREMENT PROCESSES

We now consider the amount of information which can be extracted from the results of the quantum measurement process about the observable $\hat{\chi}_S$ of the physical system in the quantum state $\hat{\rho}^S_{in}$. For this purpose, we first investigate the output probability $P^A_{out}(\nu)$ of the measurement apparatus. Substituting (2.4) into (2.7), we obtain

$$P^A_{out}(\nu) = \mathrm{Tr}_{SA}[(\hat{I}_S \otimes \hat{E}^A_{\mathcal{Y}}(\nu))\mathcal{U}_{SA}(\hat{\rho}^S_{in} \otimes \hat{\rho}^A_{in})\mathcal{U}^\dagger_{SA}]$$
$$= \mathrm{Tr}_{SA}[\mathcal{U}^\dagger_{SA}(\hat{I}_S \otimes \hat{E}^A_{\mathcal{Y}}(\nu))\mathcal{U}_{SA}(\hat{\rho}^S_{in} \otimes \hat{\rho}^A_{in})]$$
$$= \mathrm{Tr}_S[\hat{\Pi}_S(\nu)\hat{\rho}^S_{in}] \tag{3.1}$$

where the operator $\hat{\Pi}_S(\nu)$ of the physical system is given by

$$\hat{\Pi}_S(\nu) = \mathrm{Tr}_A[\mathcal{U}^\dagger_{SA}(\hat{I}_S \otimes \hat{E}^A_{\mathcal{Y}}(\nu))\mathcal{U}_{SA}(\hat{I}_S \otimes \hat{\rho}^A_{in})] \tag{3.2}$$

Using the properties of the POVM $\hat{E}^A_{\mathcal{Y}}(\nu)$ of the measurement apparatus, it is easily seen that the operator $\hat{\Pi}_S(\nu)$ becomes the POVM of the physical system, which satisfies the relations

$$\hat{\Pi}_S(\nu) \geq 0, \qquad \sum_{\nu \in \mathcal{N}} \hat{\Pi}_S(\nu) = \hat{I}_S \tag{3.3}$$

Furthermore, using the completeness relation of the eigenstate $|\psi_S(\mu)\rangle$ of the observable $\hat{\chi}_S$, we can calculate (3.1) as follows:

$$\begin{aligned}
P^A_{\text{out}}(\nu) &= \sum_{\mu\in\mathcal{M}} \langle\psi_S(\mu)|\hat{\Pi}_S(\nu)\hat{\rho}^S_{\text{in}}|\psi_S(\mu)\rangle \\
&= \sum_{\mu\in\mathcal{M}} \sum_{\mu'\in\mathcal{M}} \langle\psi_S(\mu)|\hat{\Pi}_S(\nu)|\psi_S(\mu')\rangle\langle\psi_S(\mu')|\hat{\rho}^S_{\text{in}}|\psi_S(\mu)\rangle \\
&= \sum_{\mu\in\mathcal{M}} \langle\psi_S(\mu)|\hat{\Pi}_S(\nu)|\psi_S(\mu)\rangle\langle\psi_S(\mu)|\hat{\rho}^S_{\text{in}}|\psi_S(\mu)\rangle + \mathcal{R}(\nu) \qquad (3.4)
\end{aligned}$$

where the quantity $\mathcal{R}(\nu)$ is given by

$$\mathcal{R}(\nu) = \sum_{\substack{\mu\in\mathcal{M},\,\mu'\in\mathcal{M}\\(\mu\neq\mu')}}\!\!\sum \langle\psi_S(\mu)|\hat{\Pi}_S(\nu)|\psi_S(\mu')\rangle\,\langle\psi_S(\mu')|\hat{\rho}^S_{\text{in}}|\psi_S(\mu)\rangle \qquad (3.5)$$

It should be noted that the first term on the right-hand side of equation (3.4) includes only the diagonal elements with respect to the eigenstates of the observable $\hat{\chi}_S$, while the second term includes only the off-diagonal elements.

If we can prepare the physical system in one of the eigenstates of the observable $\hat{\chi}_S$, e.g., $\hat{\rho}^S_{\text{in}} = |\psi_S(\mu)\rangle\,\langle\psi_S(\mu)|$, the output probability of the measurement apparatus becomes

$$P_{SA}(\nu|\mu) = \langle\psi_S(\mu)|\,\hat{\Pi}_S(\nu)|\psi_S(\mu)\rangle \qquad (3.6)$$

This indicates that when we repeatedly perform the quantum measurement on identically prepared physical systems, we can obtain the probabilities $P^A_{\text{out}}(\nu)$ and $P_{SA}(\nu|\mu)$ from the measurement outcomes. On the other hand, the quantity $\mathcal{R}(\nu)$ cannot be determined by the quantum measurement process characterized by the triplet $\mathbb{M} = \langle\hat{\rho}^A_{\text{in}}, \hat{E}^A_{\mathcal{Y}}(\nu), \mathcal{U}_{SA}\rangle$. Therefore we assume that the quantum measurement process satisfies the relation (Fine, 1969)

$$\langle\psi_S(\mu)|\hat{\Pi}_S(\nu)|\psi_S(\mu')\rangle = 0 \qquad (\mu \neq \mu') \qquad (3.7)$$

The physical meaning of this relation will be considered later.

When the quantum measurement process $\mathbb{M} = \langle\hat{\rho}^A{}_{\text{in}}, \hat{E}^A_{\mathcal{Y}}(\nu), \mathcal{U}_{SA}\rangle$ satisfies relation (3.7), the output probability $P^A_{\text{out}}(\nu)$ of the measurement apparatus becomes

$$P^A_{\text{out}}(\nu) = \sum_{\mu\in\mathcal{M}} P_{SA}(\nu|\mu)P^S_{\text{in}}(\mu) \qquad (3.8)$$

where $P_{SA}(\nu|\mu)$ is given by equation (3.6). Since the operator $\hat{\Pi}_S(\nu)$ is the POVM of the physical system, $P_{SA}(\nu|\mu)$ satisfies the relations

$$P_{SA}(\nu|\mu) \geq 0, \qquad \sum_{\nu\in\mathcal{N}} P_{SA}(\nu|\mu) = 1 \qquad (3.9)$$

It is easy to see from equations (3.8) and (3.9) that $P_{SA}(\nu|\mu)$ represents the conditional probability that the measurement outcome of the quantum measurement process is given by the value ν when the observable $\hat{\chi}_S$ takes the value μ in the premeasurement state $\hat{\rho}^S_{\text{in}}$ of the physical system. According to the Bayes theorem (Caves and Drummond, 1994), we obtain the joint probability $P_{SA}(\nu, \mu)$ and the *posterior* probability $P_{AS}(\mu|\nu)$ in the quantum measurement process,

$$P_{AS}(\mu|\nu) = \frac{P_{SA}(\nu|\mu)P^S_{\text{in}}(\mu)}{P^A_{\text{out}}(\nu)} \tag{3.10}$$

$$P_{SA}(\nu, \mu) = P_{SA}(\nu|\mu)P^S_{\text{in}}(\mu) = P_{AS}(\mu|\nu)P^A_{\text{out}}(\nu) \tag{3.11}$$

It can be considered that the information about the observable $\hat{\chi}_S$ of the physical system which is extracted from the measurement outcomes is equivalent to the information transmitted from the physical system in the premeasurement state to the measurement apparatus in the output state. This means that the quantum measurement process characterized by the triplet $\mathbb{M} = \langle \hat{\rho}^A_{\text{in}}, \hat{E}^A_{\mathcal{Y}}(\nu), \mathcal{U}_{SA}\rangle$ defines a communication channel between the measured physical system and the measurement apparatus. Therefore we can express the amount of information about the observable $\hat{\chi}_S$ of the physical system extracted from the measurement outcomes by the mutual information between the physical system and the measurement apparatus,

$$\begin{aligned} I(Y^A_{\text{out}}; X^S_{\text{in}}) &= H(X^S_{\text{in}}) - H(X^S_{\text{in}}|Y^A_{\text{out}}) \\ &= H(X^S_{\text{in}}) + H(Y^A_{\text{out}}) - H(Y^A_{\text{out}}, X^S_{\text{in}}) \\ &= H(Y^A_{\text{out}}) - H(Y^A_{\text{out}}|X^S_{\text{in}}) \\ &= I(X^S_{\text{in}}; Y^A_{\text{out}}) \end{aligned} \tag{3.12}$$

where the entropies $H(X^S_{\text{in}})$ and $H(Y^A_{\text{out}})$ are given by equations (2.18) and (2.20), and the joint entropy $H(Y^A_{\text{out}}, X^S_{\text{in}})$ and the conditional entropies $H(X^S_{\text{in}}|Y^A_{\text{out}})$ and $H(Y^A_{\text{out}}|X^S_{\text{in}})$ are given respectively by

$$H(Y^A_{\text{out}}, X^S_{\text{in}}) = -\sum_{\mu\in\mathcal{M}}\sum_{\nu\in\mathcal{N}} P_{SA}(\nu, \mu) \ln P_{SA}(\nu, \mu) \tag{3.13}$$

$$H(X^S_{\text{in}}|Y^A_{\text{out}}) = -\sum_{\mu\in\mathcal{M}}\sum_{\nu\in\mathcal{N}} P_{SA}(\nu, \mu) \ln P_{AS}(\mu|\nu) \tag{3.14}$$

$$H(Y^A_{\text{out}}|X^S_{\text{in}}) = -\sum_{\mu\in\mathcal{M}}\sum_{\nu\in\mathcal{N}} P_{SA}(\nu, \mu) \ln P_{SA}(\nu|\mu) \tag{3.15}$$

Because of equations (3.10) and (3.11), these entropies satisfy the relation

$$\begin{aligned} H(Y^A_{\text{out}}, X^S_{\text{in}}) &= H(Y^A_{\text{out}}) + H(X^S_{\text{in}}|Y^A_{\text{out}}) \\ &= H(X^S_{\text{in}}) + H(Y^A_{\text{out}}|X^S_{\text{in}}) \end{aligned} \tag{3.16}$$

Comparing equation (2.16) with equation (3.8) and using equations (3.12)–(3.15), we find that under our assumption [see (3.7)], the probability reproducibility condition is equivalent to the entropic relation given by $H(X^S_{\text{in}}|Y^A_{\text{out}}) = H(Y^A_{\text{out}}|X^S_{\text{in}}) = 0$, or equivalently $I(Y^A_{\text{out}};\, X^S_{\text{in}}) = H(X^S_{\text{in}}) = H(Y^A_{\text{out}})$. In Section 4 the information gain $I(Y^A_{\text{out}};\, X^S_{\text{in}})$ is compared with the entropy decrease of the physical system caused by the quantum measurement process.

We now consider the physical meaning of relation (3.7). Since the relation indicates that the operators $\hat{\Pi}_S(\nu)$ and $\hat{E}^S_\chi(\mu)$ are simultaneously diagonalized, we obtain the commutation relation

$$[\hat{\Pi}_S(\nu),\, \hat{E}^S_\chi(\mu)] = 0 \tag{3.17}$$

Here let us introduce an operator $\hat{\chi}^{\text{op}}_S(n)$ of the physical system,

$$\begin{aligned}\hat{\chi}^{\text{op}}_S(n) &= \sum_{\nu\in\mathcal{N}} \nu^n \hat{\Pi}_S(\nu) \\ &= \text{Tr}_A[\mathcal{U}^\dagger_{SA}(\hat{I}_S \otimes \mathcal{Y}_A(n))\mathcal{U}_{SA}(\hat{I}_S \otimes \hat{\rho}^A_{\text{in}})]\end{aligned} \tag{3.18}$$

where the operator $\mathcal{Y}_A(n)$ of the measurement apparatus is given by

$$\mathcal{Y}_A(n) = \sum_{\nu\in\mathcal{N}} \nu^n \hat{E}^A_y(\nu) \tag{3.19}$$

When $\hat{E}^A_y(\nu)$ is a projection operator $|\phi_A(\nu)\rangle\langle\phi_A(\nu)|$ onto the eigenspace of the pointer observable, we have $\mathcal{Y}_A(n) = \mathcal{Y}^n_A$, where $\mathcal{Y}_A = \Sigma_{\nu\in\mathcal{N}}\, \nu|\phi_A(\nu)\rangle\langle\phi_A(\nu)|$ is the spectral decomposition of the pointer observable of the measurement apparatus. It should be noted that the operator $\hat{\chi}^{\text{op}}_S(n)$ depends only on the quantum measurement process characterized by the triplet $\mathbb{M} = \langle\hat{\rho}^A_{\text{in}},\, \hat{E}^A_{\mathcal{Y}}(\nu), \mathcal{U}_{SA}\rangle$, but is independent of the observable $\hat{\chi}_S$ of the measured physical system. Hence the operator $\hat{\chi}^{\text{op}}_S(n)$ is called the operational observable (Englert and Wódkiewicz, 1995; Banaszek and Wódkiewicz, 1997; Ban, 1997c), which is not a Hermitian operator in general, while the Hermitian operator $\hat{\chi}_S$ is called the intrinsic observable of the physical system and is independent of the quantum measurement process. The operational observable is also referred to as the fuzzy observable or the unsharp observable (Busch *et al.*, 1995; Prugovečki, 1976a, b). Using the intrinsic and operational observables, relation (3.17) can be expressed as

$$[\hat{\chi}^m_S,\, \hat{\chi}^{\text{op}}_S(n)] = 0 \tag{3.20}$$

The condition represented by (3.7) is equivalent to the commutativity of the intrinsic and operational observables in the quantum measurement process. Therefore we can summarize the result in the following form.

Theorem 3.1. If the operational observable $\hat{\chi}^{\text{op}}_S(n)$ defined by the quantum measurement process $\mathbb{M} = \langle\hat{\rho}^A_{\text{in}},\, \hat{E}^A_{\mathcal{Y}}(\nu), \mathcal{U}_{SA}\rangle$ commutes with the intrinsic observable $\hat{\chi}_S$ of the physical system, the amount of information $I(Y^A_{\rm out};\, X^S_{\rm in})$ about the intrinsic observable $\hat{\chi}_S$ extracted from the measurement outcomes can be given by the mutual information,

$$I(Y^A_{\rm out};\, X^S_{\rm in}) = \sum_{\mu\in\mathcal{M}} \sum_{\nu\in\mathcal{N}} P_{SA}(\nu|\mu) P^S_{\rm in}(\mu) \ln \left[\frac{P_{SA}(\nu|\mu)}{P^A_{\rm out}(\nu)} \right] \tag{3.21}$$

where the probabilities $P^A_{\rm out}(\nu)$, $P^S_{\rm in}(\mu)$, and $P_{SA}(\nu|\mu)$ are given respectively by equations (2.7), (2.17), and (3.6).

Next let us rewrite the condition given by (3.17) or (3.20) into another form which is used in Sections 4 and 5. Using the completeness relation of the eigenstate $|\psi_S(\mu)\rangle$ of the observable $\hat{\chi}_S$, we can express the unitary operator $\mathcal{U}_{SA}$ as

$$\mathcal{U}_{SA} = \sum_{\mu\in\mathcal{M}} \sum_{\mu'\in\mathcal{M}} |\psi_S(\mu)\rangle \hat{U}_A(\mu, \mu') \langle\psi_S(\mu')| \tag{3.22}$$

$$\mathcal{U}^\dagger_{SA} = \sum_{\mu\in\mathcal{M}} \sum_{\mu'\in\mathcal{M}} |\psi_S(\mu)\rangle \hat{U}^\dagger_A(\mu, \mu') \langle\psi_S(\mu')| \tag{3.23}$$

where the operators $\hat{U}_A(\mu, \mu')$ and $\hat{U}^\dagger_A(\mu, \mu')$ of the measurement apparatus are given by

$$\hat{U}_A(\mu, \mu') = \langle\psi_S(\mu)|\mathcal{U}_{SA}|\psi_S(\mu')\rangle \tag{3.24}$$

$$\hat{U}^\dagger_A(\mu, \mu') = \langle\psi_S(\mu)|\mathcal{U}^\dagger_{SA}|\psi_S(\mu')\rangle \tag{3.25}$$

Since the operator $\mathcal{U}_{SA}$ is unitary, these operators satisfy the relation

$$\sum_{\mu''\in\mathcal{M}} \hat{U}_A(\mu, \mu'')\, \hat{U}^\dagger_A(\mu'', \mu') = \sum_{\mu''\in\mathcal{M}} \hat{U}^\dagger_A(\mu, \mu'') \hat{U}_A(\mu'', \mu') = \delta_{\mu,\mu'} \hat{I}_A \tag{3.26}$$

When we substitute equations (3.22) and (3.23) into equation (3.2), the POVM $\hat{\Pi}_S(\nu)$ of the physical system is expressed as

$$\hat{\Pi}_S(\nu) = \sum_{\mu\in\mathcal{M}} \sum_{\mu'\in\mathcal{M}} \sum_{\mu''\in\mathcal{M}} |\psi_S(\mu)\rangle\, {\rm Tr}_A[\hat{U}^\dagger_A(\mu, \mu') \hat{E}^A_{\mathcal{Y}}(\nu) \hat{U}_A(\mu', \mu'') \hat{\rho}^A_{\rm in}] \langle\psi_S(\mu'')| \tag{3.27}$$

Therefore it is found from this equation that the condition given by (3.17) or (3.20) can be stated in the following form.

Condition 3.1. The quantum measurement process which is characterized by the triplet $\mathbb{M} = \langle\hat{\rho}^A_{\rm in},\, \hat{E}^A_{\mathcal{Y}}(\nu),\, \mathcal{U}_{SA}\rangle$ satisfies the relation

$$\sum_{\mu'\in\mathcal{M}} {\rm Tr}_A[\hat{U}^\dagger_A(\mu, \mu') \hat{E}^A_{\mathcal{Y}}(\nu) \hat{U}_A(\mu', \mu'') \hat{\rho}^A_{\rm in}] = \delta_{\mu,\mu''} \sum_{\mu'\in M} {\rm Tr}_A[\hat{U}^\dagger_A(\mu, \mu') \hat{E}^A_{\mathcal{Y}}(\nu) \hat{U}_A(\mu', \mu) \hat{\rho}^A_{\rm in}] \tag{3.28}$$

When this condition is satisfied, the POVM $\hat{\Pi}_S(\nu)$ of the physical system and the conditional probability $P_{SA}(\nu|\mu)$ in the quantum measurement process are expressed as

$$\hat{\Pi}_S(\nu) = \sum_{\mu\in\mathcal{M}} \sum_{\mu'\in\mathcal{M}} |\psi_S(\mu)\rangle \, \mathrm{Tr}_A[\hat{U}_A^\dagger(\mu, \mu')\hat{E}_{\mathcal{Y}}^A(\nu)\hat{U}_A(\mu', \mu)\hat{\rho}_{\mathrm{in}}^A]\langle\psi_S(\mu)| \quad (3.29)$$

$$P_{SA}(\nu|\mu) = \sum_{\mu'\in\mathcal{M}} \mathrm{Tr}_A[\hat{U}_A^\dagger(\mu, \mu')\hat{E}_{\mathcal{Y}}^A(\nu)\hat{U}_A(\mu', \mu)\hat{\rho}_{\mathrm{in}}^A] \quad (3.30)$$

In Sections 4, 5, and 7, we will use equations (3.28)–(3.30) to investigate the relation between the information gain and the entropy change in the quantum measurement process.

4. ENTROPY CHANGE OF A PHYSICAL SYSTEM

In this section we investigate the decrease of the Shannon entropy of a physical system in the quantum measurement process characterized by the triplet $\mathbb{M} = \langle\hat{\rho}_{\mathrm{in}}^A, \hat{E}_{\mathcal{Y}}^A(\nu), \mathcal{U}_{SA}\rangle$. When the outcome ν of the quantum measurement process is obtained, the probability $P_{\mathrm{out}}^S(\mu|\nu)$ that the observable $\hat{\chi}_S$ takes the value μ in the postmeasurement state $\hat{\rho}_{\mathrm{out}}^S(\nu)$ of the physical system is given by (2.8). The measurement outcome ν is obtained with the probability $P_{\mathrm{out}}^A(\nu)$. Thus the entropy of the physical system in the postmeasurement state is given by

$$H(X_{\mathrm{out}}^S|Y_{\mathrm{out}}^A) = -\sum_{\nu\in\mathcal{N}} \sum_{\mu\in\mathcal{M}} P_{\mathrm{out}}^A(\nu)P_{\mathrm{out}}^S(\mu|\nu) \ln P_{\mathrm{out}}^S(\mu|\nu) \quad (4.1)$$

Then the decrease of the entropy of the physical system that is caused by the quantum measurement process is calculated by

$$\begin{aligned}\Delta H(X_{\mathrm{out}}^S, X_{\mathrm{in}}^S|Y_{\mathrm{out}}^A) &= H(X_{\mathrm{in}}^S) - H(X_{\mathrm{out}}^S|Y_{\mathrm{out}}^A)\\ &= H(X_{\mathrm{in}}^S) + H(Y_{\mathrm{out}}^A) - H(X_{\mathrm{out}}^S, Y_{\mathrm{out}}^A) \quad (4.2)\end{aligned}$$

where the entropies $H(X_{\mathrm{in}}^S)$, $H(Y_{\mathrm{out}}^A)$, and $H(X_{\mathrm{out}}^S, Y_{\mathrm{out}}^A)$ are given respectively by equations (2.18), (2.20), and (2.21). Using (2.12), (3.22), and (3.23), we can calculate the joint probability $P_{\mathrm{out}}^{SA}(\mu,\nu)$ as follows:

$$\begin{aligned}P_{\mathrm{out}}^{SA}(\mu,\nu) &= \mathrm{Tr}_{SA}[(\hat{E}_\chi^S(\mu) \otimes \hat{E}_{\mathcal{Y}}^A(\nu))\mathcal{U}_{SA}(\hat{\rho}_{\mathrm{in}}^S \otimes \hat{\rho}_{\mathrm{in}}^A)\mathcal{U}_{SA}^\dagger]\\ &= \mathrm{Tr}_A\langle\psi_S(\mu)|(\hat{I}_S \otimes \hat{E}_{\mathcal{Y}}^A(\nu))\mathcal{U}_{SA}(\hat{\rho}_{\mathrm{in}}^S \otimes \hat{\rho}_{\mathrm{in}}^A)\mathcal{U}_{SA}^\dagger|\psi_S(\mu)\rangle\\ &= \sum_{\mu'\in\mathcal{M}} \sum_{\mu''\in\mathcal{M}} \mathrm{Tr}_A[\hat{U}_A^\dagger(\mu', \mu)\hat{E}_{\mathcal{Y}}^A(\nu)\hat{U}_A(\mu, \mu'')\hat{\rho}_{\mathrm{in}}^A]\langle\psi_S(\mu'')|\hat{\rho}_{\mathrm{in}}^S|\psi_S(\mu')\rangle\end{aligned} \quad (4.3)$$

To proceed further, we impose the following condition on the quantum measurement process $\mathbb{M} = \langle\hat{\rho}_{\mathrm{in}}^A, \hat{E}_{\mathcal{Y}}^A(\nu), \mathcal{U}_{SA}\rangle$.

Condition 4.1. The quantum measurement process which is characterized by the triplet $\mathbb{M} = \langle \hat{\rho}^A_{\text{in}}, \hat{E}^A_{\mathcal{Y}}(\nu), \mathcal{U}_{SA} \rangle$ satisfies the relation

$$\begin{aligned}&\text{Tr}_A[\hat{U}^\dagger_A(\mu, \mu')\hat{E}^A_{\mathcal{Y}}(\nu)\hat{U}_A(\mu', \mu'')\hat{\rho}^A_{\text{in}}] \\ &\quad = \delta_{\mu, \mu''} \text{Tr}_A[\hat{U}^\dagger_A(\mu, \mu')\hat{F}^A_{\mathcal{Y}}(\nu)\hat{U}_A(\mu', \mu)\hat{\rho}^A_{\text{in}}] \end{aligned} \tag{4.4}$$

It is easy to see that this condition is stronger than Condition 3.1. In fact, when Condition 4.1 is satisfied by the quantum measurement process, Condition 3.1 is always fulfilled. Under Condition 4.1, the joint probability $P^{SA}_{\text{out}}(\mu, \nu)$ becomes

$$P^{SA}_{\text{out}}(\mu, \nu) = \sum_{\mu' \in \mathcal{M}} \text{Tr}_A[\hat{U}^\dagger_A(\mu', \mu)\hat{E}^A_{\mathcal{Y}}(\nu)\hat{U}_A(\mu, \mu')\hat{\rho}^A_{\text{in}}]P^S_{\text{in}}(\mu') \tag{4.5}$$

Here we further impose the following condition on the quantum measurement process.

Condition 4.2. The quantum measurement process which is characterized by the triplet $\mathbb{M} = \langle \hat{\rho}^A_{\text{in}}, \hat{E}^A_{\mathcal{Y}}(\nu), \mathcal{U}_{SA} \rangle$ satisfies the relation

$$\begin{aligned}&\text{Tr}_A[\hat{U}^\dagger_A(\mu', \mu)\hat{E}^A_{\mathcal{Y}}(\nu)\hat{U}_A(\mu, \mu')\hat{\rho}^A_{\text{in}}] \\ &\quad = \delta_{\mu', f(\mu; \nu)} \text{Tr}_A[\hat{U}^\dagger_A(f(\mu; \nu), \mu)\, \hat{E}^A_{\mathcal{Y}}(\nu)\hat{U}_A(\mu, f(\mu; \nu))\hat{\rho}^A_{\text{in}}] \end{aligned} \tag{4.6}$$

where $f(\mu; \nu) \in \mathcal{M}$ is a function of μ that in general depends on the outcome ν of the quantum measurement process.

Furthermore, if $f(\mu; \nu) \neq \mu$, we introduce the following condition.

Condition 4.3. The conditional probability $P_{SA}(\nu|\mu)$ given by (3.6) or (3.30) and the spectral set $\mathcal{M}$ of the observable $\hat{\chi}_S$ satisfy the relation

$$\sum_{\mu \in \mathcal{M}} P_{SA}(\nu|f(\mu; \nu))F(f(\mu; \nu)) = \sum_{\mu \in \mathcal{M}} P_{SA}(\nu|\mu)F(\mu) \tag{4.7}$$

for any nonsingular function $F(\mu)$.

Using equations (3.24) and (3.25), we find that Conditions 4.1 and 4.2 can be unified in the following relation:

$$\text{Tr}_A[\mathcal{U}^\dagger_{SA}(\hat{E}^S_\chi(\mu) \otimes \hat{E}^A_{\mathcal{Y}}(\nu))\mathcal{U}_{SA}(\hat{I}_S \otimes \hat{\rho}^A_{\text{in}})] = P_{SA}(\nu|f(\mu; \nu))\hat{E}^S_\chi(f(\mu; \nu)) \tag{4.8}$$

When the quantum measurement process satisfies Conditions 4.1–4.3, we can obtain the joint probability from (4.5),

$$P^{SA}_{\text{out}}(\mu, \nu) = P_{SA}(\nu|f(\mu; \nu))P^S_{\text{in}}(f(\mu; \nu)) \tag{4.9}$$

where we have used the fact that under Condition 4.2, the conditional probability $P_{SA}(\nu|\mu)$ given by (3.30) becomes

$$P_{SA}(\nu|\mu) = \mathrm{Tr}_A[\hat{U}_A^{\dagger}(\mu, f^{-1}(\mu;\nu))\, \hat{E}_{\mathcal{Y}}^{A}(\nu)\hat{U}_A(f^{-1}(\mu;\nu), \mu)\, \hat{\rho}_{\mathrm{in}}^{A}] \quad (4.10)$$

where $f^{-1}(\mu;\nu)$ is an inverse of the function $f(\mu;\nu)$. In addition, if the function $f(\mu;\nu)$ does not depend on the measurement outcome ν, namely $f(\mu;\nu) = f(\mu)$, taking the summation of (4.9) with respect to ν yields the equality

$$P_{\mathrm{out}}^{S}(\mu) = P_{\mathrm{in}}^{S}(f(\mu)) \quad (4.11)$$

where the probability $P_{\mathrm{out}}^{S}(\mu)$ is given by equation (2.10). We have found that under Conditions 4.1–4.3, the joint probability $P_{\mathrm{out}}^{SA}(\mu, \nu)$ is greatly simplified. We will see that there are many quantum measurement processes that satisfy these conditions (see Sections 5–7).

Finally when the quantum measurement process satisfies Conditions 4.1–4.3, we can obtain the decrease of the entropy $\Delta H(X_{\mathrm{out}}^{S}, X_{\mathrm{in}}^{S}|Y_{\mathrm{out}}^{A})$ of the physical system in the quantum measurement process,

$$\begin{aligned}
&\Delta H(X_{\mathrm{out}}^{S}, X_{\mathrm{in}}^{S}|Y_{\mathrm{out}}^{A}) \\
&\quad = H(X_{\mathrm{in}}^{S}) + H(Y_{\mathrm{out}}^{A}) + \sum_{\mu\in\mathcal{M}}\sum_{\nu\in\mathcal{N}} P_{SA}(\mu,\nu)\ln P_{SA}(\mu,\nu) \\
&\quad = H(X_{\mathrm{in}}^{S}) + H(Y_{\mathrm{out}}^{A}) \\
&\quad + \sum_{\mu\in\mathcal{M}}\sum_{\nu\in\mathcal{N}} P_{SA}(\nu|f(\mu;\nu))\, P_{\mathrm{in}}^{S}(f(\mu;\nu))\ln[P_{SA}(\nu|f(\mu;\nu))P_{\mathrm{in}}^{S}(f(\mu;\nu))] \\
&\quad = H(X_{\mathrm{in}}^{S}) + H(Y_{\mathrm{out}}^{A}) + \sum_{\mu\in\mathcal{M}}\sum_{\nu\in\mathcal{N}} P_{SA}(\nu|\mu)P_{\mathrm{in}}^{S}(\mu)\ln[P_{SA}(\nu|\mu)P_{\mathrm{in}}^{S}(\mu)] \\
&\quad = H(Y_{\mathrm{out}}^{A}) + \sum_{\mu\in\mathcal{M}}\sum_{\nu\in\mathcal{N}} P_{SA}(\nu|\mu)\, P_{\mathrm{in}}^{S}(\mu)\ln P_{SA}(\nu|\mu) \\
&\quad = H(Y_{\mathrm{out}}^{A}) - H(Y_{\mathrm{out}}^{A}|X_{\mathrm{in}}^{S}) \qquad (4.12)
\end{aligned}$$

Comparing this result with equation (3.12), we find that the entropy decrease of the physical system caused by the quantum measurement process is equal to the amount of information about the observable $\hat{\chi}_S$ of the physical system that can be extracted from the measurement outcomes. Thus we obtain the equality

$$I(Y_{\mathrm{out}}^{A}; X_{\mathrm{in}}^{S}) = \Delta H(X_{\mathrm{out}}^{S}, X_{\mathrm{in}}^{S}|Y_{\mathrm{out}}^{A}) \quad (4.13)$$

Furthermore, substituting (3.12) and (4.2) into this equation, we obtain the relation between the conditional entropies,

$$H(X^S_{\rm in}|Y^A_{\rm out}) = H(X^S_{\rm out}|Y^A_{\rm out}) \tag{4.14}$$

This result indicates that when we obtain the measurement outcome, the uncertainty of the observable $\hat{\chi}_S$ in the premeasurement state of the physical system is equal to that in the postmeasurement state. Therefore we can summarize the results in the following theorem.

Theorem 4.1. When the quantum measurement process $\mathbb{M} = \langle\hat{\rho}^A_{\rm in}, \hat{E}^A_{\mathcal{Y}}(\nu), \mathcal{U}_{SA}\rangle$ satisfies Conditions 4.1–4.3, the entropy decrease of the physical system is equal to the amount of information extracted from the measurement outcomes,

$$I(Y^A_{\rm out};\, X^S_{\rm in}) = \Delta H(X^S_{\rm out},\, X^S_{\rm in}|Y^A_{\rm out})$$

In this case, the the equality $H(X^S_{\rm in}|Y^A_{\rm out}) = H(X^S_{\rm out}|Y^A_{\rm out})$ is also established, which indicates that although the quantum state of the physical system changes due to the quantum measurement process, the uncertainty of the observable $\hat{\chi}_S$ in the premeasurement state is equal to that in the postmeasurement state when the measurement outcome is obtained.

It is important to note that Conditions 4.1–4.3 are sufficient, but not necessary, for this theorem to be established.

Before closing this section, we compare the change of the Shannon entropy with that of the von Neumann entropy in the quantum measurement process $\mathbb{M} = \langle\hat{\rho}^A_{\rm in}, \hat{E}^A_{\mathcal{Y}}(\nu), \mathcal{U}_{SA}\rangle$. Suppose that the premeasurement state $\hat{\rho}^S_{\rm in}$ of the physical system is prepared in the statistical mixture of the eigenstates of the observable $\hat{\chi}_S$ which is given by

$$\hat{\rho}^S_{\rm in} = \sum_{\mu\in\mathcal{M}} P^S_{\rm in}(\mu)|\psi_S(\mu)\rangle\langle\psi_S(\mu)| \tag{4.15}$$

In this case, the von Neumann entropy $S(X^S_{\rm in})$ and the Shannon entropy $H(X^S_{\rm in})$ of the premeasurement state of the physical system are equal,

$$\begin{aligned} S(X^S_{\rm in}) &= -{\rm Tr}_S[\hat{\rho}^S_{\rm in} \ln \hat{\rho}^S_{\rm in}] \\ &= -\sum_{\mu\in\mathcal{M}} P^S_{\rm in}(\mu) \ln P^S_{\rm in}(\mu) = H(X^S_{\rm in}) \end{aligned} \tag{4.16}$$

On the other hand, in the postmeasurement state of the physical system, the von Neumann entropy is calculated to be

$$\begin{aligned} S(X^S_{\rm out}|Y^A_{\rm out}) &= -\sum_{\nu\in\mathcal{N}} P^A_{\rm out}(\nu){\rm Tr}_S[\hat{\rho}^S_{\rm out}(\nu) \ln \hat{\rho}^S_{\rm out}(\nu)] \\ &\leq -\sum_{\nu\in\mathcal{N}} P^A_{\rm out}(\nu) \sum_{\mu\in\mathcal{M}} \langle\psi_S(\mu)|\hat{\rho}^S_{\rm out}(\nu)|\psi_S(\mu)\rangle \ln \langle\psi_S(\mu)|\hat{\rho}^S_{\rm out}(\nu)|\psi_S(\mu)\rangle \end{aligned}$$

$$= -\sum_{\nu\in\mathcal{N}} P^A_{\text{out}}(\nu) \sum_{\mu\in\mathcal{M}} (\text{Tr}_S[\hat{E}^S_\chi(\mu)\hat{\rho}^S_{\text{out}}(\nu)]) \ln (\text{Tr}_S[\hat{E}^S_\chi(\mu)\hat{\rho}^S_{\text{out}}(\nu)])$$

$$= -\sum_{\nu\in\mathcal{N}} \sum_{\mu\in\mathcal{M}} P^A_{\text{out}}(\nu) P^S_{\text{out}}(\mu|\nu) \ln P^S_{\text{out}}(\mu|\nu)$$

$$= H(X^S_{\text{out}}|Y^A_{\text{out}}) \tag{4.17}$$

where we have used the concavity of the entropic function $[f(x) = -x \ln x]$ (Wehrl, 1987) or Jensen's inequality in information theory (Cover and Thomas, 1991). Using (4.16) and (4.17), we obtain the relation for the decrease of the von Neumann entropy

$$\begin{aligned}\Delta S(X^S_{\text{out}}, X^S_{\text{in}}|Y^A_{\text{out}}) &= S(X^S_{\text{in}}) - S(X^S_{\text{out}}|Y^A_{\text{out}}) \\ &\geq H(X^S_{\text{in}}) - H(X^S_{\text{out}}|Y^A_{\text{out}}) \\ &= \Delta H(X^S_{\text{out}}, X^S_{\text{in}}|Y^A_{\text{out}})\end{aligned} \tag{4.18}$$

Thus when the premeasurement state of the physical system is given by equation (4.15), the decrease of the von Neumann entropy is no less than that of the Shannon entropy.

We next consider the case that the postmeasurement state $\hat{\rho}^S_{\text{out}}(\nu)$ of the physical system is the statistical mixture of the eigenstates of the observable $\hat{\chi}_S$,

$$\hat{\rho}^S_{\text{out}}(\nu) = \sum_{\mu\in\mathcal{M}} P^S_{\text{out}}(\mu|\nu)|\psi_S(\mu)\rangle\langle\psi_S(\mu)| \tag{4.19}$$

Then it is easily seen that the following equality holds:

$$S(X^S_{\text{out}}|Y^A_{\text{out}}) = H(X^S_{\text{out}}|Y^A_{\text{out}}) \tag{4.20}$$

Since the inequality $S(X^S_{\text{in}}) \leq H(X^S_{\text{in}})$ is satisfied in general, the decrease of the von Neumann entropy is no greater than that of the Shannon entropy,

$$\Delta S(X^S_{\text{out}}, X^S_{\text{in}}|Y^A_{\text{out}}) \leq \Delta H(X^S_{\text{out}}, X^S_{\text{in}}|Y^A_{\text{out}}) \tag{4.21}$$

Therefore we can summarize the result in the following theorem.

Theorem 4.2. The decreases of the Shannon entropy and the von Neumann entropy of the physical system in the quantum measurement processes $\mathbb{M} = \langle\hat{\rho}^A_{\text{in}}, \hat{E}^A_{\mathcal{Y}}(\nu), \mathcal{U}_{SA}\rangle$ satisfy the inequalities

$$\Delta H(X^S_{\text{out}}, X^S_{\text{in}}|Y^A_{\text{out}}) \leq \Delta S(X^S_{\text{out}}, X^S_{\text{in}}|Y^A_{\text{out}}) \qquad \text{for} \quad [\hat{\chi}_S, \hat{\rho}^S_{\text{in}}] = 0 \tag{4.22}$$

$$\Delta H(X^S_{\text{out}}, X^S_{\text{in}}|Y^A_{\text{out}}) \geq \Delta S(X^S_{\text{out}}, X^S_{\text{in}}|Y^A_{\text{out}}) \quad \text{for} \quad [\hat{\chi}_S, \hat{\rho}^S_{\text{out}}(\nu)] = 0 \tag{4.23}$$

where the equality holds for $[\hat{\chi}_S, \hat{\rho}^S_{\text{in}}] = [\hat{\chi}_S, \hat{\rho}^S_{\text{out}}(\nu)] = 0$.

We finally note that the change of the von Neumann entropy in the quantum measurement process has been investigated in detail by several authors (Groenewold, 1971; Lindblad, 1973; Ozawa, 1986).

5. EXAMPLES OF QUANTUM MEASUREMENT PROCESSES

In this section we consider several examples of quantum measurement processes to examine the general results obtained in Sections 3 and 4. In particular, we pay attention to whether Conditions 4.1–4.3 are satisfied or not by the quantum measurement processes and to whether the equality between the entropy decrease of the physical system and the amount of the information extracted from the measurement outcomes is established or not. The examples considered here include the normal unitary process (Beltrametti *et al.*, 1989) and the SU(2) and SU(1, 1) processes with photon number measurement (Ban, 1996a).

5.1. Normal Unitary Process

We first consider the normal unitary process (Beltrametti *et al.*, 1989) by means of which we obtain the information about the observable $\hat{\chi}_S$ of the physical system in the quantum state $\hat{\rho}^S_{\text{in}}$. The normal unitary process which is characterized by the triplet $\mathbb{M} = \langle \hat{\rho}^A_{\text{in}}, \hat{E}^A_{\mathcal{Y}}(\nu), \mathcal{U}_{SA}$ is set up in the following way.

1. The measurement apparatus is prepared in a pure quantum state $|\phi^A_{\text{in}}\rangle$ before the interaction with the physical system. Thus we have $\hat{\rho}^A_{\text{in}} = |\phi^A_{\text{in}}\rangle\langle\phi^A_{\text{in}}|$.
2. The readout of the measurement outcome is performed by measuring a pointer observable $\mathcal{Y}_A$ which corresponds to the same physical quantity as that represented by the intrinsic observable $\hat{\chi}_S$ of the physical system, that is, $\mathcal{Y}_A = \hat{\chi}_A$. In this case we have $\mathcal{N} = \mathcal{M}$. When we denote the eigenstate of the pointer observable $\mathcal{Y}_A$ as $|\phi_A(\nu)\rangle$, the POVM $\hat{E}^A_{\mathcal{Y}}(\nu)$ of the measurement apparatus becomes the projection operator $\hat{E}^A_{\mathcal{Y}}(\nu) = |\phi_A(\nu)\rangle\langle\phi_A(\nu)|$.
3. The unitary operator $\mathcal{U}_{SA}$ which describes the state change caused by the interaction between the physical system and the measurement apparatus is defined by the relation

$$\mathcal{U}_{SA}(|\psi_S(\mu)\rangle \otimes |\phi^A_{\text{in}}\rangle) = |\tilde{\psi}_S(\mu)\rangle \otimes |\phi_A(\mu)\rangle \tag{5.1}$$

where $|\psi_S(\mu)\rangle$ is the eigenstate of the observable $\hat{\chi}_S$ and $|\tilde{\psi}_S(\mu)\rangle$ is some quantum state of the physical system that is a nonorthogonal state in general.

In this measurement process the compound quantum state of the physical system and the measurement apparatus after the interaction becomes

$$\hat{\rho}_{\text{out}}^{SA} = \sum_{\mu\in\mathcal{M}} \sum_{\mu'\in\mathcal{M}} \langle\psi_S(\mu)|\hat{\rho}_{\text{in}}^{S}|\psi_S(\mu')\rangle|\tilde{\psi}_S(\mu)\rangle\langle\tilde{\psi}_S(\mu')| \otimes |\phi_A(\mu)\rangle\langle\phi_A(\mu')| \tag{5.2}$$

The postmeasurement state $\hat{\rho}_{\text{out}}^{S}(\nu)$ of the physical system after the measurement outcome ν was obtained and the probability $P_{\text{out}}^{A}(\nu)$ of the measurement outcome ν are calculated from (2.6), (2.7), and (5.2),

$$\hat{\rho}_{\text{out}}^{S}(\nu) = |\tilde{\psi}_S(\nu)\rangle\langle\tilde{\psi}_S(\nu)| \tag{5.3}$$

$$P_{\text{out}}^{A}(\nu) = \langle\psi_S(\nu)|\hat{\rho}_{\text{in}}^{S}|\psi_S(\nu)\rangle = P_{\text{in}}^{S}(\nu) \tag{5.4}$$

It is clear from equation (5.4) that the normal unitary process satisfies the probability reproducibility condition. Furthermore, (5.4) shows that the conditional probability $P_{SA}(\nu|\mu)$ and the *posterior* probability $P_{SA}(\mu|\nu)$ are given by

$$P_{SA}(\nu|\mu) = P_{AS}(\mu|\nu) = \delta_{\mu,\nu} \tag{5.5}$$

which indicates that $H(X_{\text{in}}^{S}|Y_{\text{out}}^{A}) = H(Y_{\text{out}}^{A}|X_{\text{in}}^{S}) = 0$. Thus the amount of information obtained from the measurement outcomes is equal to the entropies of the premeasurement state of the physical system and of the output state of the measurement apparatus,

$$I(Y_{\text{out}}^{A};\, X_{\text{in}}^{S}) = H(X_{\text{in}}^{S}) = H(Y_{\text{out}}^{A}) \tag{5.6}$$

It is seen from the definition (5.1) that the operators $\hat{U}_A(\mu,\, \mu')$ and $\hat{U}_A^{\dagger}(\mu,\, \mu')$ of the measurement apparatus [see (3.24) and (3.25)] satisfies the relation

$$\begin{aligned}&\text{Tr}_A[\hat{U}_A^{\dagger}(\mu_1,\, \mu_2)\hat{E}_{\mathcal{Y}}^{A}(\nu)\hat{U}_A(\mu_3,\, \mu_4)\hat{\rho}_{\text{in}}^{A}]\\ &\quad = \delta_{\nu,\mu_1}\delta_{\mu_1,\mu_4}\,\langle\tilde{\psi}_S(\mu_1)|\psi_S(\mu_2)\rangle\langle\psi_S(\mu_3)|\tilde{\psi}_S(\mu_1)\rangle\end{aligned} \tag{5.7}$$

which ensures that Conditions 3.1 and 4.1 are fulfilled. Then it is easy to see from (3.29) and (3.30) that $\hat{\Pi}_S(\nu) = |\psi_S(\nu)\rangle\langle\psi_S(\nu)|$ and $P_{SA}(\nu|\mu) = \delta_{\mu,\nu}$ are obtained; which is consistent with (5.5). Thus the operational observable $\hat{\chi}_S^{\text{op}}(n)$ defined by the normal unitary process [see (3.18)] is equal to the intrinsic observable $\hat{\chi}_S^{n}$ of the physical system.

In the postmeasurement state $\hat{\rho}_{\text{out}}^{S}(\nu)$ of the physical system, the observable $\hat{\chi}_S$ takes the value μ with probability $P_{\text{out}}^{S}(\mu|\nu) = |\langle\psi_S(\mu)|\tilde{\psi}_S(\nu)\rangle|^2$. Then the entropy decrease of the physical system in the normal unitary process is obtained from (4.2),

$$\begin{aligned}&\Delta H(X_{\text{out}}^{S},\, X_{\text{in}}^{S}|Y_{\text{out}}^{A})\\ &\quad = H(X_{\text{in}}^{S}) + \sum_{\mu\in\mathcal{M}} \sum_{\nu\in\mathcal{M}} P_{\text{in}}^{S}(\nu)|\langle\psi_S(\mu)|\tilde{\psi}(\nu)\rangle|^2 \ln|\langle\psi_S(\mu)|\tilde{\psi}(\nu)\rangle|^2\end{aligned} \tag{5.8}$$

Furthermore it is seen from equation (5.7) that Condition 4.2 is satisfied if and only if the quantum state $|\tilde{\psi}_S(\mu)\rangle$ of the physical system after the measurement is one of the eigenstates of the observable $\hat{\chi}_S$, that is, $|\tilde{\psi}_S(\mu)\rangle = |\psi_S(f(\mu))\rangle$ with $f(\mu) \in \mathcal{M}$. In this case, the second term on the right-hand side of (5.8) vanishes and thus the entropy decrease of the physical system becomes equal to the amount of information obtained in the normal unitary process,

$$\Delta H(X^S_{\text{out}}, X^S_{\text{in}}|Y^A_{\text{out}}) = I(Y^A_{\text{out}}; X^S_{\text{in}}) = H(X^S_{\text{in}}) = H(Y^A_{\text{out}}) \tag{5.9}$$

It is seen from equations (4.6), (5.7), and (5.8) that Condition 4.2 is necessary for the second term on the right-hand side of (5.8) to vanish. This means that Condition 4.2 is necessary and sufficient for (5.9) to be established in the normal unitary process. When the quantum state $|\tilde{\psi}_S(\mu)\rangle$ is the eigenstate of the observable $\hat{\chi}_S$, the normal unitary process is called the von Neumann–Lüders measurement (Busch *et al.*, 1991). Therefore the normal unitary process must be the von Neumann–Lüders measurement for the amount of information extracted from the measurement outcomes to equal the entropy decrease of the physical system. Since we have $[\hat{\chi}_S, \hat{\rho}^S_{\text{out}}(\nu)] = 0$ for the normal unitary process of the von Neumann–Lüders type, the decrease of the Shannon entropy is no less than that of the vou Neumann entropy (see Theorem 4.2).

5.2. SU(2) and SU(1, 1) Processes

We next consider the SU(2) and SU(1, 1) processes in quantum optical system, which are realized, respectively, by means of a lossless beam splitter and a nondegenerate parametric amplifier, to obtain the information about the photon number $\hat{\chi}_S = \hat{a}_S^\dagger \hat{a}_S$ $[\hat{E}^S_\chi(n) = |n_S\rangle\langle n_S|]$ of the physical system, where $\hat{a}_S$ and $\hat{a}_S^\dagger$ are bosonic annihilation and creation operators and $|n_S\rangle$ is the photon-number eigenstate $(\hat{a}_S^\dagger \hat{a}_S|n_S\rangle = n|n_S\rangle)$. The quantum measurement process which is characterized by the triplet $\mathbb{M} = \langle \hat{\rho}^A_{\text{in}}, \hat{E}^A_Y(n), \mathcal{U}_{SA}\rangle$ is set up in the following way.

1. The measurement apparatus is prepared in the vacuum state $\hat{\rho}^A_{\text{in}} = |0_A\rangle\langle 0_A|$ before the interaction with the physical system.
2. The readout of the measurement outcome is performed by measuring the pointer observable of the measurement apparatus, that is, the photon number operator, $\mathcal{Y}_A = \hat{a}_A^\dagger \hat{a}_A$, where $\hat{a}_A$ and $\hat{a}_A^\dagger$ are bosonic annihilation and creation operators. Thus we have $\hat{E}^A_{\mathcal{Y}}(n) = |n_A\rangle\langle n_A|$ and $\mathcal{N} = \mathcal{M} = \mathbb{N}$, where $|n_A\rangle$ is the eigenstate of the photon number operator of the measurement apparatus $(\hat{a}_A^\dagger \hat{a}_A|n_A\rangle = n|n_A\rangle)$ and $\mathbb{N}$ is the set of all nonnegative integers.
3. The unitary operator $\mathcal{U}_{SA}$ that describes the state change caused by

the interaction between the physical system and the measurement apparatus is given by

$$\hat{\mathcal{U}}_{SA} = \exp[-\theta(\hat{J}_+^{SA} - \hat{J}_-^{SA})] \qquad \text{for the SU(2) process} \tag{5.10}$$

$$\hat{\mathcal{U}}_{SA} = \exp[\theta(\hat{K}_+^{SA} - \hat{K}_-^{SA})] \qquad \text{for the SU(1, 1) process} \tag{5.11}$$

where $\hat{J}_\pm^{SA}$ and $\hat{J}_0^{SA}$ are the generators of the SU(2) Lie algebra and $\hat{K}_\pm^{SA}$ and $\hat{K}_0^{SA}$ are the generators of the SU(1, 1) Lie algebra,

$$\hat{J}_+^{SA} = \hat{a}_S^\dagger \hat{a}_A , \quad \hat{J}_-^{SA} = \hat{a}_S \hat{a}_A^\dagger , \quad \hat{J}_0^{SA} = \frac{1}{2}(\hat{a}_S^\dagger \hat{a}_S - \hat{a}_A^\dagger \hat{a}_A) \tag{5.12}$$

$$\hat{K}_+^{SA} = \hat{a}_S^\dagger \hat{a}_A^\dagger, \quad \hat{K}_-^{SA} = \hat{a}_S \hat{a}_A, \quad \hat{K}_0^{SA} = \frac{1}{2}(\hat{a}_S^\dagger \hat{a}_S + \hat{a}_A^\dagger \hat{a}_A + 1) \tag{5.13}$$

which satisfy the SU(2) and SU(1, 1) Lie commutation relations,

$$[\hat{J}_+^{SA}, \hat{J}_-^{SA}] = 2\hat{J}_0^{SA} , \qquad [\hat{J}_0^{SA}, \hat{J}_\pm^{SA}] = \pm\hat{J}_\pm^{SA} \tag{5.14}$$

$$[\hat{K}_-^{SA}, \hat{K}_+^{SA}] = 2\hat{K}_0^{SA}, \qquad [\hat{K}_0^{SA}, \hat{K}_\pm^{SA}] = \pm\hat{K}_\pm^{SA} \tag{5.15}$$

In the following, we first consider the SU(2) process and then the SU(1, 1) process.

5.2.1. SU(2) Process

In the SU(2) process, the compound quantum state of the physical system and the measurement apparatus after the interaction is calculated from (2.4) and (5.10) (Ban, 1994, 1996a),

$$\hat{\rho}_{\text{out}}^{SA} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[\frac{1}{m!n!} \left(\frac{\mathcal{R}}{\mathcal{T}} \right)^{m+n} \right]^{1/2} \hat{a}_S^m \, \mathcal{T}^{1/2\hat{a}_S^\dagger \hat{a}_S} \, \hat{\rho}_{\text{in}}^S \, \mathcal{T}^{1/2\hat{a}_S^\dagger \hat{a}_S} \, \hat{a}_S^{\dagger n} \otimes |m_A\rangle\langle n_A| \tag{5.16}$$

where $\mathcal{T} = \cos^2\theta$ and $\mathcal{R} = \sin^2\theta$ are the transmittance and reflectance of the lossless beam splitter. The quantum state $\hat{\rho}_{\text{out}}^S(m)$ of the physical system after the measurement outcome m was obtained and the probability $P_{\text{out}}^A(m)$ of the measurement outcome m are given respectively by

$$\hat{\rho}_{\text{out}}^S(m) = \frac{\hat{a}_S^m \, \mathcal{T}^{1/2\hat{a}_S^\dagger \hat{a}_S} \, \hat{\rho}_{\text{in}}^S \, \mathcal{T}^{1/2\hat{a}_S^\dagger \hat{a}_S} \, \hat{a}_S^{\dagger m}}{\text{Tr}_S[\hat{a}_S^m \, \mathcal{T}^{1/2\hat{a}_S^\dagger \hat{a}_S} \, \hat{\rho}_{\text{in}}^S \, \mathcal{T}^{1/2\hat{a}_S^\dagger \hat{a}_S} \, \hat{a}_S^{\dagger m}]} \tag{5.17}$$

$$P_{\text{out}}^A(m) = \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!} \mathcal{R}^m \mathcal{T}^{n-m} \langle n_S | \hat{\rho}_{\text{in}}^S | n_S \rangle \tag{5.18}$$

These results are equivalent to those obtained for the continuous measurement of photon number that obeys the quantum Markov process (Ban, 1994). The matrix element of the unitary operator $\mathcal{U}_{SA}$ given by (5.10) is calculated to be

$$\langle m_A, n_S|\mathcal{U}_{SA}|n'_S, 0_A\rangle = \langle m_A|\hat{U}_A(n, n')|0_A\rangle = \sqrt{F(m, n')}\delta_{n,n'-m} \quad (5.19)$$

with

$$F(m, n) = \frac{n!}{m!(n-m)!}\mathcal{R}^m\mathcal{T}^{n-m} \quad (5.20)$$

Using equation (5.19), we obtain the following relation:

$$\begin{aligned}&\mathrm{Tr}_A[\hat{U}_A^\dagger(n_1, n_2)\hat{E}_{\mathcal{Y}}^A(m)\hat{U}_A(n_3, n_4)\hat{\rho}_{\mathrm{in}}^A]\\&\quad = \sqrt{F(m, n_1)F(m, n_4)}\delta_{n_2,n_1-m}\delta_{n_3,n_4-m}\end{aligned} \quad (5.21)$$

It is easy to see from this relation that Conditions 3.1 and 4.1–4.3 with $f(n; m) = n + m$ are fulfilled in the SU(2) process with the photon number measurement. Therefore the entropy decrease of the physical system is equal to the amount of information extracted from the measurement outcomes (see Theorem 4.1).

Substituting equation (5.21) into equation (3.29), we obtain the POVM $\hat{\Pi}_S(m)$ of the physical system in the SU(2) process with the photon number measurement,

$$\hat{\Pi}_S(m) = \sum_{n=0}^{\infty} F(m, n)|n_S\rangle\langle n_S| \quad (5.22)$$

which indicates that the conditional probability is given by $P_{SA}(m|n) = \langle n_S|\hat{\Pi}_S(m)|n_S\rangle = F(m, n)$. This is consistent with (5.18). Then the operational observable $\hat{\chi}_S^{\mathrm{op}}(m)$ of the system defined by the SU(2) process with the photon number measurement is given by

$$\mathcal{N}_S^{\mathrm{op}}(n) = \sum_{m=0}^{\infty} m^n\hat{\Pi}_S(m) = \frac{\partial^n}{\partial\xi^n}\mathcal{G}_S(\xi)\bigg|_{\xi=0} \quad (5.23)$$

with

$$\mathcal{G}_S(\xi) = [1 + \mathcal{R}(e^\xi - 1)]^{\hat{a}_S^\dagger\hat{a}_S} \quad (5.24)$$

In particular, we obtain for $n = 1$ and $n = 2$,

$$\mathcal{N}_S^{\mathrm{op}}(1) = \mathcal{R}\hat{a}_S^\dagger\hat{a}_S, \qquad \mathcal{N}_S^{\mathrm{op}}(2) = (\mathcal{R}\hat{a}_S^\dagger\hat{a}_S)^2 + \mathcal{R}(1 - \mathcal{R})\hat{a}_S^\dagger\hat{a}_S \quad (5.25)$$

which clearly shows that $\mathcal{N}_S^{\mathrm{op}}(n) \neq [\mathcal{N}_S^{\mathrm{op}}(1)]^n$. It is obvious from equations (5.23) and (5.24) that the operational observable $\mathcal{N}_S^{\mathrm{op}}(n)$ commutes with the intrinsic observable $\mathcal{N}_S = \hat{a}_S^\dagger \hat{a}_S$. Furthermore, it is easily seen from (5.17) that if the premeasurement state $\hat{\rho}_{\text{in}}^S$ is diagonal with respect to the photon-number eigenstate $|n_S\rangle$, the postmeasurement state $\hat{\rho}_{\text{out}}^S(m)$ of the physical system is also diagonal. In this case, the decrease of the Shannon entropy becomes equal to that of the von Neumann entropy (see Theorem 4.2).

Finally we show by explicit calculation the equality between the entropy decrease of the physical system and the amount of information extracted from the measurement outcomes. The amount of information obtained from the results of the SU(2) process with the photon number measurement is given by

$$I(Y_{\text{out}}^A;\, X_{\text{in}}^S) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} P_{SA}(m|n) P_{\text{in}}^S(n) \ln \left[\frac{P_{SA}(m|n)}{P_{\text{out}}^A(m)} \right] \tag{5.26}$$

where $P_{\text{in}}^S(n) = \langle n_S|\hat{\rho}_S|n_S\rangle$. On the other hand, the entropy decrease in the measurement process is given by

$$\Delta H(X_{\text{out}}^S,\, X_{\text{in}}^S|Y_{\text{out}}^A) = -\sum_{n=0}^{\infty} P_{\text{in}}^S(n) \ln P_{\text{in}}^S(n) + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_{\text{out}}^A(m) \langle n_S|\hat{\rho}_{\text{out}}^S(m)|n_S\rangle \ln\langle n_S|\hat{\rho}_{\text{out}}^S(m)|n_S\rangle \tag{5.27}$$

Since from (5.17) we obtain

$$\langle n_S|\hat{\rho}_{\text{out}}^S(m)|n_S\rangle = \frac{P_{SA}(m|n+m) P_{\text{in}}^S(n+m)}{\sum_{n=0}^{\infty} P_{SA}(m|n+m) P_{\text{in}}^S(n+m)} = \frac{P_{SA}(m|n+m) P_{\text{in}}^S(n+m)}{P_{\text{out}}^A(m)} \tag{5.28}$$

we can calculate the entropy decrease $\Delta H(X_{\text{out}}^S,\, X_{\text{in}}^S|Y_{\text{out}}^A)$ as follows:

$$\begin{aligned}\Delta H(X_{\text{out}}^S,\, X_{\text{in}}^S|Y_{\text{out}}^A) &= -\sum_{n=0}^{\infty} P_{\text{in}}^S(n) \ln P_{\text{in}}^S(n) \\ &\quad + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} P_{SA}(m|n+m) P_{\text{in}}^S(n+m) \\ &\quad \times \ln \left[\frac{P_{SA}(m|n+m) P_{\text{in}}^S(n+m)}{P_{\text{out}}^A(m)} \right] \\ &= -\sum_{n=0}^{\infty} P_{\text{in}}^S(n) \ln P_{\text{in}}^S(n)\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} P_{SA}(m|n) P_{\text{in}}^{S}(n) \ln \left[\frac{P_{SA}(m|n) P_{\text{in}}^{S}(n)}{P_{\text{out}}^{A}(m)} \right] \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} P_{SA}(m|n) P_{\text{in}}^{S}(n) \ln \left[\frac{P_{SA}(m|n)}{P_{\text{out}}^{A}(m)} \right] \\
&= I(Y_{\text{out}}^{A}; X_{\text{in}}^{S})
\end{aligned} \tag{5.29}
$$

where we have used the relation

$$
\begin{aligned}
\sum_{n=0}^{\infty} P_{SA}(m|n+m) P_{\text{in}}^{S}(n+m) &= \sum_{n=m}^{\infty} P_{SA}(m|n) P_{\text{in}}^{S}(n) \\
&= \sum_{n=0}^{\infty} P_{SA}(m|n) P_{\text{in}}^{S}(n)
\end{aligned} \tag{5.30}
$$

which indicates that Condition 4.3 is fulfilled. Therefore we have shown by explicit calculation the equality $\Delta H(X_{\text{out}}^{S}, X_{\text{in}}^{S} | Y_{\text{out}}^{A}) = I(Y_{\text{out}}^{A}; X_{\text{in}}^{S})$ in the SU(2) process with the photon number measurement.

5.2.2. SU(1, 1) Process

We next consider the SU(1, 1) process with the photon number measurement. From (2.4) and (5.11) we obtain the compound quantum state of the physical system and the measurement apparatus after the interaction (Ban, 1994, 1997a),

$$
\hat{\rho}_{\text{out}}^{SA} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\mathcal{L}^{1/2(m+n)}}{\sqrt{m!n!}} \hat{a}_S^{\dagger m} \mathcal{K}^{1/2 \hat{a}_S \hat{a}_S^{\dagger}} \hat{\rho}_{\text{in}}^{S} \mathcal{K}^{1/2 \hat{a}_S \hat{a}_S^{\dagger}} \hat{a}_S^{n} \otimes |m_A\rangle\langle n_A| \tag{5.31}
$$

where we have defined the parameters $\mathcal{L} = \tanh^2 \theta$ and $\mathcal{K} = 1/\cosh^2\theta$ ($\mathcal{L} + \mathcal{K} = 1$). The postmeasurement state $\hat{\rho}_{\text{out}}^{S}(m)$ of the physical system after the measurement outcome m was obtained and the probability $P_{\text{out}}^{A}(m)$ of the measurement outcome m are given respectively by

$$
\hat{\rho}_{\text{out}}^{S}(m) = \frac{\hat{a}_S^{\dagger m} \mathcal{K}^{1/2 \hat{a}_S \hat{a}_S^{\dagger}} \hat{\rho}_{\text{in}}^{S} \mathcal{K}^{1/2 \hat{a}_S \hat{a}_S^{\dagger}} \hat{a}_S^{m}}{\mathrm{Tr}_S[\hat{a}_S^{\dagger m} \mathcal{K}^{1/2 \hat{a}_S \hat{a}_S^{\dagger}} \hat{\rho}_{\text{in}}^{S} \mathcal{K}^{1/2 \hat{a}_S \hat{a}_S^{\dagger}} \hat{a}_S^{m}]} \tag{5.32}
$$

$$
P_{\text{out}}^{A}(m) = \sum_{n=0}^{\infty} \frac{(n+m)!}{m!n!} \mathcal{L}^{m} \mathcal{K}^{n+1} \langle n_S | \hat{\rho}_{\text{in}}^{S} | n_S \rangle \tag{5.33}
$$

Comparing these equations with (5.16) and (5.17), we find the similarity between the SU(2) and SU(1, 1) processes with the photon number measurement. If we exchange the annihilation (creation) operators with the creation (annihilation) operators, we obtain the results for the SU(2) [SU(1, 1)] process from the SU(1, 1) [SU(2)] process. The matrix element of the unitary operator $\mathcal{U}_{SA}$ given by (5.11) is calculated to be

$$\langle m_A, n_S|\mathcal{U}_{SA}|n'_S, 0_A\rangle = \langle m_A|\hat{U}_A(n, n')|0_A\rangle = \sqrt{G(m, n')}\delta_{n,n'+m} \tag{5.34}$$

with

$$G(m, n) = \frac{(n+m)!}{m!n!}\mathcal{L}^m\mathcal{K}^{n+1} \tag{5.35}$$

which yields the relation

$$\begin{aligned}&\mathrm{Tr}_A[\hat{U}_A^\dagger(n_1, n_2)\hat{E}_{\mathcal{Y}}^A(m)\hat{U}_A(n_3, n_4)\hat{\rho}_{\rm in}^A]\\ &\quad = \sqrt{G(m, n_1)G(m, n_4)}\delta_{n_2,n_1+m}\delta_{n_3,n_4+m}\end{aligned} \tag{5.36}$$

Thus we find from this relation that Conditions 3.1 and 4.1–4.3 with $f(n; m) = n - m$ are fulfilled in the SU(1, 1) process with the photon number measurement. Therefore the entropy decrease of the physical system is equal to the amount of information obtained from the measurement outcomes.

Substituting (5.36) into (3.29), we obtain the POVM $\hat{\Pi}_S(m)$ of the physical system in the SU(1, 1) process with the photon counting measurement,

$$\hat{\Pi}_S(m) = \sum_{n=0}^{\infty} G(m, n)|n_S\rangle\langle n_S| \tag{5.37}$$

which indicates that the conditional probability is given by $P_{SA}(m|n) = G(m, n)$. This result is consistent with (5.33). The operational observable $\hat{\chi}_S^{\rm op}(m)$ of the physical system defined by the SU(1, 1) process with the photon number measurement is given by

$$\mathcal{N}_S^{op}(n) = \sum_{m=0}^{\infty} m^n\hat{\Pi}_S(m) = \frac{\partial^n}{\partial\xi^n}\mathcal{G}_S(\xi)\bigg|_{\xi=0} \tag{5.38}$$

with

$$\mathcal{G}_S(\xi) = \left(\frac{\mathcal{K}}{1-\mathcal{L}e^\xi}\right)^{\hat{a}_S\hat{a}_S^\dagger} \tag{5.39}$$

In particular, we obtain for $n = 1$ and $n = 2$,

$$\mathcal{N}_S^{\rm op}(1) = \frac{\mathcal{L}}{\mathcal{K}}\hat{a}_S\hat{a}_S^\dagger, \qquad \mathcal{N}_S^{\rm op}(2) = \left(\frac{\mathcal{L}}{\mathcal{K}}\hat{a}_S\hat{a}_S^\dagger\right)^2 + \frac{\mathcal{L}}{\mathcal{K}}\left(1+\frac{\mathcal{L}}{\mathcal{K}}\right)\hat{a}_S\hat{a}_S^\dagger \tag{5.40}$$

where the parameter $\mathcal{L}/\mathcal{K} = \sinh^2\theta$ represents the enhanced vacuum fluctuation caused by the nondegenerate parametric amplifier (Walls and Milburn, 1994). Furthermore, as we have seen in the case of the SU(2) process, the decrease of the Shannon entropy becomes equal to that of the von Neumann entropy if the premeasurement state $\hat{\rho}^S_{\text{in}}$ of the physical system is diagonal with respect to the photon-number eigenstate $|n_S\rangle$.

Before closing this section, we show by explicit calculation the equality between the entropy decrease of the physical system and the amount of information obtained from the measurement outcomes. The amount of information obtained from the measurement outcomes is given by (5.26) and the entropy decrease of the physical system in the measurement process is given by (5.27). In the case of the SU(1, 1) process, instead of (5.28), we obtain from (5.32)

$$\begin{aligned}\langle n_S|\hat{\rho}^S_{\text{out}}(m)|n_S\rangle &= \frac{P_{SA}(m|n-m)P^S_{\text{in}}(n-m)}{\sum_{n=0}^{\infty} P_{SA}(m|n-m)\, P^S_{\text{in}}(n-m)} \\ &= \frac{P_{SA}(m|n-m)P^S_{\text{in}}(n-m)}{P^A_{\text{out}}(m)}\end{aligned} \tag{5.41}$$

Thus we can calculate the entropy decrease $\Delta H(X^S_{\text{out}}, X^S_{\text{in}}|Y^A_{\text{out}})$ as follows:

$$\begin{aligned}\Delta H(X^S_{\text{out}}, X^S_{\text{in}}|Y^A_{\text{out}}) &= -\sum_{n=0}^{\infty} P^S_{\text{in}}(n) \ln P^S_{\text{in}}(n) \\ &\quad + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_{SA}(m|n-m)P^S_{\text{in}}(n-m) \\ &\quad \times \ln\left[\frac{P_{SA}(m|n-m)P^S_{\text{in}}(n-m)}{P^A_{\text{out}}(m)}\right] \\ &= -\sum_{n=0}^{\infty} P^S_{\text{in}}(n) \ln P^S_{\text{in}}(n) \\ &\quad + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_{SA}(m|n)P^S_{\text{in}}(n) \ln\left[\frac{P_{SA}(m|n)P^S_{\text{in}}(n)}{P^A_{\text{out}}(m)}\right] \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_{SA}(m|n)P^S_{\text{in}}(n) \ln\left[\frac{P_{SA}(m|n)}{P^A_{\text{out}}(m)}\right] \\ &= I(Y^A_{\text{out}}; X^S_{\text{in}})\end{aligned} \tag{5.42}$$

where we have used the relation

$$\begin{aligned}\sum_{n=0}^{\infty} P_{SA}(m|n)P^S_{\text{in}}(n) &= \sum_{n=m}^{\infty} P_{SA}(m|n-m)P^S_{\text{in}}(n-m) \\ &= \sum_{n=0}^{\infty} P_{SA}(m|n-m)P^S_{\text{in}}(n-m)\end{aligned} \tag{5.43}$$

which means that the SU(1, 1) process with the photon number measurement satisfies Condition 4.3. Therefore we have shown by explicit calculation the equality $\Delta H(X^S_{\text{out}}, X^S_{\text{in}}|Y^A_{\text{out}}) = I(Y^A_{\text{out}}; X^S_{\text{in}})$ in the SU(1, 1) process with the photon number measurement.

6. QUANTUM MEASUREMENT OF A CONTINUOUS OBSERVABLE

We have investigated the information about an intrinsic observable of a physical system extracted from the measurement outcomes and the entropy change of the measured physical system caused by the quantum measurement processes, where we have assumed that the observables have a discrete spectrum (discrete observable). In this section, we will consider the information gain and the entropy change in quantum measurement processes of observables which have a continuous spectrum (continuous observables). The mathematically rigorous treatment of quantum measurement processes of continuous observables has been formulated by Ozawa (1984). In this section we will treat them in a physically acceptable way, though the treatment is not mathematically rigorous.

6.1. Information Gain and Entropy Change

We first reformulate the results obtained in Sections 3 and 4 to consider the information gain and entropy change in quantum measurement processes of continuous observables. Suppose that we perform some quantum measurement on a physical system to obtain the information about an observable $\hat{\chi}_S$ which is expanded in the following form:

$$\hat{\chi}_S = \int_{\mu\in\mathcal{M}} d\mu\ \mu|\psi_S(\mu)\rangle\langle\psi_S(\mu)| = \int_{\mu\in\mathcal{M}} d\mu\ \mu\hat{E}^S_\chi(\mu) \tag{6.1}$$

where $\hat{E}^S_\chi(\mu) = |\psi_S(\mu)\rangle\langle\psi_S(\mu)|$ is a projection operator onto the eigenspace of the observable $\hat{\chi}_S$ and $\mathcal{M}$ represents the spectral set. We assume that the eigenstate $|\psi_S(\mu)\rangle$ of the observable $\hat{\chi}_S$ satisfies the relations,

$$\langle\psi_S(\mu_1)|\psi_S(\mu_2)\rangle = \delta(\mu_1 - \mu_2), \qquad \int_{\mu\in\mathcal{M}} d\mu\ |\psi_S(\mu)\rangle\langle\psi_S(\mu)| = \hat{I}_S \tag{6.2}$$

The readout of the measurement outcome whose value belongs to an infinitesimal interval $[\nu, \nu + d\nu)$ is described by the POVM $\hat{E}^A_{\mathcal{Y}}(\nu)\ d\nu$ of the measurement apparatus, where the operator $\hat{E}^A_{\mathcal{Y}}(\nu)$ satisfies

$$\hat{E}^A_{\mathcal{Y}}(\nu) \geq 0, \qquad \int_{\nu\in\mathcal{N}} d\nu\ \hat{E}^A_{\mathcal{Y}}(\nu) = \hat{I}_A \tag{6.3}$$

Here $\mathcal{N}$ represents the set of all possible outcomes of the quantum measurement process. If the measurement outcome ν is obtained by measuring the value of a continuous pointer observable $\mathcal{Y}_A$ of the measurement apparatus, the POVM $\hat{E}^A_{\mathcal{Y}}(\nu)\,d\nu$ becomes a projection operator $\hat{E}^A_{\mathcal{Y}}(\nu)\,d\nu = |\phi_A(\nu)\rangle\langle\phi_A(\nu)|d\nu$, where $|\phi_A(\nu)\rangle$ is the eigenstate of the pointer observable $\mathcal{Y}_A$. The quantum measurement process of the continuous observable is characterized by the triplet $\mathbb{M} = \langle\hat{\rho}^A_{\text{in}}, \hat{E}^A_{\mathcal{Y}}(\nu), \mathcal{U}_{SA}\rangle$, where $\hat{\rho}^A_{\text{in}}$ is the initial quantum state of the measurement apparatus and $\mathcal{U}_{SA}$ is the unitary operator which describes the state change caused by the interaction between the physical system and the measurement apparatus.

When we obtain the value ν as the result of the quantum measurement process, the postmeasurement state $\hat{\rho}^S_{\text{out}}(\nu)$ of the physical system is given by the reduction formula (2.6), and the probability density $P^A_{\text{out}}(\nu)$ that the measurement outcome ν is obtained is given by (2.7), where the normalization condition is modified to be $\int_{\nu\in\mathcal{N}} d\nu\, P^A_{\text{out}}(\nu) = 1$. Relations (2.8)–(2.10), (2.13)–(2.17), and (2.15) are still valid for quantum measurement processes of continuous observables if the probabilities appearing there are replaced with the probability densities. Relations (2.11) and (2.14) are modified as follows:

$$P^S_{\text{out}}(\mu) = \int_{\nu\in\mathcal{N}} d\nu\, P^S_{\text{out}}(\mu|\nu) P^A_{\text{out}}(\nu) \tag{6.4}$$

$$P^S_{\text{out}}(\mu) = \int_{\nu\in\mathcal{N}} d\nu\, P^{SA}_{\text{out}}(\mu, \nu), \qquad P^A_{\text{out}}(\nu) = \int_{\nu\in\mathcal{M}} d\mu\, P^{SA}_{\text{out}}(\mu, \nu) \tag{6.5}$$

As we have done in the quantum measurement process of the discrete observable, we can introduce continuous random variables in the quantum measurement process of the continuous observable. For continuous random variables, the Shannon entropy is called the differential entropy (Cover and Thomas, 1991). Then we obtain the differential entropies in the quantum measurement process of the continuous observable,

$$H(X^S_{\text{in}}) = -\int_{\mu\in\mathcal{M}} d\mu\, P^S_{\text{in}}(\mu) \ln P^S_{\text{in}}(\mu) \tag{6.6}$$

$$H(X^S_{\text{out}}) = -\int_{\mu\in\mathcal{M}} d\mu\, P^S_{\text{out}}(\mu) \ln P^S_{\text{out}}(\mu) \tag{6.7}$$

$$H(Y^A_{\text{out}}) = -\int_{\nu\in\mathcal{N}} d\nu\, P^A_{\text{out}}(\nu) \ln P^A_{\text{out}}(\nu) \tag{6.8}$$

$$H(X^S_{\text{out}}, Y^A_{\text{out}}) = -\int_{\mu\in\mathcal{M}} d\mu \int_{\nu\in\mathcal{N}} d\nu\, P^{SA}_{\text{out}}(\mu, \nu) \ln P^{SA}_{\text{out}}(\mu, \nu) \tag{6.9}$$

Furthermore, the conditional entropies are given by

$$H(X^S_{\text{out}}|Y^A_{\text{out}}) = -\int_{\mu\in\mathcal{M}} d\mu \int_{\nu\in\mathcal{N}} d\nu\, P^{SA}_{\text{out}}(\mu, \nu) \ln P^S_{\text{out}}(\mu|\nu) \quad (6.10)$$

$$H(Y^A_{\text{out}}|X^S_{\text{out}}) = -\int_{\mu\in\mathcal{M}} d\mu \int_{\nu\in\mathcal{N}} d\nu\, P^{SA}_{\text{out}}(\mu, \nu) \ln P^A_{\text{out}}(\nu|\mu) \quad (6.11)$$

The relations among the entropies given by (2.24) and (2.25) are valid for the quantum measurement process of the continuous observable. It should be noted that the differential entropy can take negative values (Cover and Thomas, 1991).

The output probability density $P^A_{\text{out}}(\nu)$ of the measurement apparatus can be expressed in the following form:

$$P^A_{\text{out}}(\nu) = \text{Tr}_A[\hat{\Pi}_S(\nu)\hat{\rho}^S_{\text{in}}] \quad (6.12)$$

where $\hat{\Pi}_S(\nu)\, d\nu$ is the POVM of the physical system and the operator $\hat{\Pi}_S(\nu)$ is given by (3.2), and satisfies the relations

$$\hat{\Pi}_S(\nu) \geq 0, \qquad \int_{\nu\in\mathcal{N}} d\nu\, \hat{\Pi}_S(\nu) = \hat{I}_S \quad (6.13)$$

For the quantum measurement process of the continuous observable, we assume that the POVM $\hat{\Pi}_S(\nu)\, d\nu$ of the physical system satisfies the relation

$$\langle\psi_S(\mu)|\hat{\Pi}_S(\nu)|\psi_S(\mu')\rangle = \delta(\mu - \mu')P_{SA}(\nu|\mu) \quad (6.14)$$

which is equivalent to the condition given by (3.7). In this equation $P_{SA}(\nu|\mu)$ represents the conditional probability density that the measurement outcome ν is obtained when the observable $\hat{\chi}_S$ of the physical system takes the value μ in the premeasurement state $\hat{\rho}^S_{\text{in}}$. When this condition is satisfied, the POVM $\hat{\Pi}_S(\nu)\, d\nu$ of the physical system and the output probability density $P^A_{\text{out}}(\nu)$ of the measurement apparatus become

$$\hat{\Pi}_S(\nu) = \int_{\mu\in\mathcal{M}} d\mu\, |\psi_S(\mu)\rangle P_{SA}(\nu|\mu)\langle\psi_S(\mu)| \quad (6.15)$$

$$P^A_{\text{out}}(\nu) = \int_{\mu\in\mathcal{M}} d\mu\, P_{SA}(\nu|\mu)P^S_{\text{in}}(\mu) \quad (6.16)$$

where $P^S_{\text{in}}(\mu) = \langle\psi_S(\mu)|\hat{\rho}^S_{\text{in}}|\psi_S(\mu)\rangle$ is the probability density that the observable $\hat{\chi}_S$ takes the value μ in the premeasurement state $\hat{\rho}^S_{\text{in}}$ of the physical system.

According to the Bayes theorem (Caves and Drummond, 1994), we obtain the joint probability density $P_{SA}(\nu, \mu)$ and the *posterior* probability density $P_{AS}(\mu|\nu)$, which are given respectively by (3.10) and (3.11). The amount of information $I(Y^A_{\text{out}}; X^S_{\text{in}})$ about the observable $\hat{\chi}_S$ of the physical system extracted from the measurement outcomes is given by (3.12)–(3.15) in which the summation is replaced with the integration,

$$H(Y^A_{\text{out}}, X^S_{\text{in}}) = -\int_{\mu\in\mathcal{M}} d\mu \int_{\nu\in\mathcal{N}} d\nu\; P_{SA}(\nu, \mu) \ln P_{SA}(\nu, \mu) \quad (6.17)$$

$$H(X^S_{\text{in}}|Y^A_{\text{out}}) = -\int_{\mu\in\mathcal{M}} d\mu \int_{\nu\in\mathcal{N}} d\nu\; P_{SA}(\nu, \mu) \ln P_{AS}(\mu|\nu) \quad (6.18)$$

$$H(Y^A_{\text{out}}|X^S_{\text{in}}) = -\int_{\mu\in\mathcal{M}} d\mu \int_{\nu\in\mathcal{N}} d\nu\; P_{SA}(\nu, \mu) \ln P_{SA}(\nu|\mu) \quad (6.19)$$

which satisfy relation (3.16).

The operational observable $\hat{\chi}^{\text{op}}_S(n)$ of the physical system defined by the quantum measurement process of the continuous observable $\hat{\chi}_S$ is given by

$$\begin{aligned}\hat{\chi}^{\text{op}}_S(n) &= \int_{\nu\in\mathcal{N}} d\nu\; \nu^n \hat{\Pi}_S(\nu)\\ &= \text{Tr}_A[\mathcal{U}^\dagger_{SA}(\hat{I}_S \otimes \mathcal{Y}_A(n))\mathcal{U}_{SA}(\hat{I}_S \otimes \hat{\rho}^A_{\text{in}})] \end{aligned}\quad (6.20)$$

where the operator $\mathcal{Y}_A(n)$ of the measurement apparatus is defined by

$$\mathcal{Y}_A(n) = \int_{\nu\in\mathcal{N}} d\nu\; \nu^n \hat{E}^A_{\mathcal{Y}}(\nu) \quad (6.21)$$

which becomes the spectral decomposition of the pointer observable if $\hat{E}^A_{\mathcal{Y}}(\nu)$ is the projection operator. Using the operational and intrinsic observables of the physical system, the condition given by (6.14) is expressed as the commutation relation $[\hat{\chi}^{\text{op}}_S(n), \hat{\chi}_S] = 0$. Therefore we can obtain the following theorem.

Theorem 6.1. If the operational observable $\hat{\chi}^{\text{op}}_S(n)$ defined by the quantum measurement process $\mathbb{M} = \langle\hat{\rho}^A_{\text{in}}, \hat{E}^A_{\mathcal{Y}}(\nu), \mathcal{U}_{SA}\rangle$ commutes with the intrinsic observable $\hat{\chi}_S$ of the physical system, the amount of information $I(Y^A_{\text{out}}; X^S_{\text{in}})$ about the intrinsic observable $\hat{\chi}_S$ extracted from the measurement outcomes can be expressed by the Shannon mutual information,

$$I(Y^A_{\text{out}}; X^S_{\text{in}}) = \int_{\mu\in\mathcal{M}} d\mu \int_{\nu\in\mathcal{N}} d\nu\; P_{SA}(\nu|\mu) P^S_{\text{in}}(\mu) \ln\left[\frac{P_{SA}(\nu|\mu)}{P^A_{\text{out}}(\nu)}\right] \quad (6.22)$$

where the probability densities $P_{SA}(\nu|\mu)$ and $P^{A}_{\text{out}}(\nu)$ are given by (6.14) and (6.16).

To investigate the relation between the information gain and the entropy change in the quantum measurement process of the continuous observable, let us express the unitary operator $\mathcal{U}_{SA}$ in the following form:

$$\mathcal{U}_{SA} = \int_{\mu \in \mathcal{M}} d\mu \int_{\mu' \in \mathcal{M}} d\mu' \, |\psi_S(\mu)\rangle \hat{U}_A(\mu, \mu') \langle \psi_S(\mu')| \tag{6.23}$$

$$\mathcal{U}^{\dagger}_{SA} = \int_{\mu \in \mathcal{M}} d\mu \int_{\mu' \in \mathcal{M}} d\mu' \, |\psi_S(\mu)\rangle \hat{U}^{\dagger}_A(\mu, \mu') \langle \psi_S(\mu')| \tag{6.24}$$

where the operators $\hat{U}_A(\mu, \mu')$ and $\hat{U}^{\dagger}_A(\mu, \mu')$ of the measurement apparatus are given by (3.24) and (3.25). Since the operator $\hat{U}_{SA}$ is unitary, the operators $\hat{U}_A(\mu, \mu')$ and $\hat{U}^{\dagger}_A(\mu, \mu')$ satisfies the relation

$$\int_{\mu'' \in \mathcal{M}} d\mu'' \, \hat{U}_A(\mu, \mu'') \hat{U}^{\dagger}_A(\mu'', \mu') = \int_{\mu'' \in \mathcal{M}} d\mu'' \, \hat{U}^{\dagger}_A(\mu, \mu'') \hat{U}_A(\mu'', \mu') = \delta(\mu - \mu') \hat{I}_A \tag{6.25}$$

In terms of the operators $\hat{U}_A(\mu, \mu')$ and $\hat{U}^{\dagger}_A(\mu, \mu')$, the condition given by (6.14) is expressed in the following form.

Condition 6.1. The quantum measurement process which is characterized by the triplet $\mathbb{M} = \langle \hat{\rho}^{A}_{\text{in}}, \hat{E}^{A}_{\mathcal{Y}}(\nu), \mathcal{U}_{SA} \rangle$ satisfies the relation

$$\int_{\mu'' \in \mathcal{M}} d\mu'' \, \mathrm{Tr}_A[\hat{U}^{\dagger}_A(\mu, \mu'') \hat{E}^{A}_{\mathcal{Y}}(\nu) \hat{U}_A(\mu'', \mu') \hat{\rho}^{A}_{in}] = \delta(\mu - \mu') P_{SA}(\nu|\mu) \tag{6.26}$$

To proceed further, we impose the following condition on the quantum measurement process.

Condition 6.2 The quantum measurement process which is characterized by the triplet $\mathbb{M} = \langle \hat{\rho}^{A}_{\text{in}}, \hat{E}^{A}_{\mathcal{Y}}(\nu), \mathcal{U}_{SA} \rangle$ should satisfy the relation

$$\mathrm{Tr}_A[\hat{U}^{\dagger}_A(\mu', \mu) \, \hat{E}^{A}_{\mathcal{Y}}(\nu) \hat{U}_A(\mu, \mu'') \hat{\rho}^{A}_{\text{in}}] = \delta(\mu' - \mu'') \delta(\mu' - f(\mu; \nu)) P_{SA}(\nu|f(\mu; \nu)) \tag{6.27}$$

where $f(\mu;\nu) \in \mathcal{M}$ is a function of μ that in general depends on ν. If $f(\mu;\nu) \neq \mu$, the conditional probability density $P_{SA}(\nu|\mu)$ and spectral set $\mathcal{M}$ satisfy the relation,

$$\int_{\mu\in\mathcal{M}} d\mu\ P_{SA}(\nu|f(\mu;\nu))F(f(\mu;\nu)) = \int_{\mu\in\mathcal{M}} d\mu\ P_{SA}(\nu|\mu)F(\mu) \quad (6.28)$$

where $F(\mu)$ is any analytic function.

It should be noted that relation (6.27) can be expressed in the same form as that given by (4.8).

When the quantum measurement process satisfies Condition 6.2, the joint probability density $P^{SA}_{\text{out}}(\mu,\nu)$ in the compound quantum state of the physical system and the measurement apparatus after the interaction is greatly simplified to be

$$P^{SA}_{\text{out}}(\mu,\nu) = P_{SA}(\nu|f(\mu;\nu))P^{S}_{\text{in}}(f(\mu;\nu)) \quad (6.29)$$

Using this result, we can calculate the joint entropy of the physical system and the measurement apparatus as follows:

$$\begin{aligned}
&H(X^S_{\text{out}}, Y^A_{\text{out}}) \\
&= -\int_{\mu\in\mathcal{M}} d\mu \int_{\nu\in\mathcal{N}} d\nu\ P^{SA}_{\text{out}}(\mu,\nu) \ln P^{SA}_{\text{out}}(\mu,\nu) \\
&= -\int_{\mu\in\mathcal{M}} d\mu \int_{\nu\in\mathcal{N}} d\nu\ P_{SA}(\nu|f(\mu;\nu))P^{S}_{\text{in}}(f(\mu;\nu)) \\
&\quad \times \ln [P_{SA}(\nu|f(\mu;\nu))P^{S}_{\text{in}}(f(\mu;\nu))] \\
&= -\int_{\mu\in\mathcal{M}} d\mu \int_{\nu\in\mathcal{N}} d\nu\ P_{SA}(\nu|\mu)P^{S}_{\text{in}}(\mu) \ln[P_{SA}(\nu|\mu)P^{S}_{\text{in}}(\mu)] \\
&= H(X^S_{\text{in}}) - \int_{\mu\in\mathcal{M}} d\mu \int_{\nu\in\mathcal{N}} d\nu\ P_{SA}(\nu|\mu)P^{S}_{\text{in}}(\mu) \ln P_{SA}(\nu|\mu) \\
&= H(X^S_{\text{in}}) + H(Y^A_{\text{out}}) - \int_{\mu\in\mathcal{M}} d\mu \int_{\nu\in\mathcal{N}} d\nu\ P_{SA}(\nu|\mu)\ P^{S}_{\text{in}}(\mu) \ln\left[\frac{P_{SA}(\nu|\mu)}{P^{A}_{\text{out}}(\nu)}\right] \\
&= H(X^S_{\text{in}}) + H(Y^A_{\text{out}}) - I(Y^A_{\text{out}}; X^S_{\text{in}})
\end{aligned}$$

where we have used (6.16) and (6.28). When we obtain the measurement outcome, the decrease of the entropy of the physical system, $\Delta H(X^S_{\text{out}}, X^S_{\text{in}}|Y^A_{\text{out}})$, is calculated from (4.2). Therefore we can obtain the following theorem.

Theorem 6.2. When the quantum measurement process of the continuous observable $\mathbb{M} = \langle \hat{\rho}^A_{\rm in}, \hat{E}^A_{\mathcal{Y}}(\nu), \mathcal{U}_{SA} \rangle$ satisfies Condition 6.2, the entropy decrease of the physical system is equal to the amount of information that can be extracted from the measurement outcomes,

$$I(Y^A_{\rm out}; X^S_{\rm in}) = \Delta H(X^S_{\rm out}, X^S_{\rm in} | Y^A_{\rm out}) \tag{6.31}$$

In this case, the equality of the conditional entropies $H(X^S_{\rm in}|Y^A_{\rm out}) = H(X^S_{\rm out}|Y^A_{\rm out})$ also holds, which indicates that the uncertainty of the observable $\hat{X}_S$ in the premeasurement state is equal to that in the postmeasurement state when the measurement outcome is obtained.

It should be noted that Condition 6.2 is sufficient, but not necessary, for this theorem to be established.

6.2. Position and Momentum Measurements

We now consider the quantum measurement process of the canonical position observable of a physical system in one-dimensional space to examine the general results obtained above. Let $\hat{x}_S$ be the canonical position operator of the measured physical system and let $|\hat{x}_S\rangle$ be the position eigenstate such that $\hat{x}_S|x_S\rangle = x|x_S\rangle$. Then we have the projection operator $\hat{E}^S_{\chi}(x) = |x_S\rangle\langle x_S|$ and the spectral set $\mathcal{M} = \mathbb{R}$, where $\mathbb{R}$ stands for the set of all real numbers. The quantum measurement process $\mathbb{M} = \langle \hat{\rho}^A_{\rm in}, \hat{E}^A_{\mathcal{Y}}(\nu), \hat{U}_{SA} \rangle$ of the position observable of the physical system is set up in the following way.

1. The measurement apparatus of the position measurement is prepared in an arbitrary quantum state $\hat{\rho}^A_{\rm in}$ before the interaction with the physical system. The measurement accuracy of position, of course, depends on this quantum state.
2. The readout of the measurement outcome is performed by measuring the pointer observable, which is the position operator $\hat{x}_A$ of the measurement apparatus. Thus we have the projection operator $\hat{E}^A_{\mathcal{Y}}(x) = |x_A\rangle\langle x_A|$ and the spectral set $\mathcal{N} = \mathbb{R}$, where $|x_A\rangle$ is the position eigenstate of the measurement apparatus, such that $\hat{x}_A|x_A\rangle = x|x_A\rangle$.
3. The unitary operator $\mathcal{U}_{SA}$ that describes the state change caused by the interaction between the physical system and the measurement apparatus is given by

$$\mathcal{U}_{SA} = \exp(-i\hat{x}_S\hat{p}_A) \tag{6.32}$$

where $\hat{p}_A$ is the momentum operator of the measurement apparatus that is canonical conjugate to the position operator $\hat{x}_A$. In this equation, we set $\hbar = 1$ and $g\tau = 1$, for the sake of simplicity, where the parameter g stands for the coupling constant.

In this measurement process, the compound quantum state of the physical system and the measurement apparatus after the interaction becomes

$$\hat{\rho}^{SA}_{\text{out}} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dy' \; \langle x_S|\hat{\rho}^S_{\text{in}}|x'_S\rangle\langle y_A|\hat{\rho}^A_{\text{in}}|y'_A\rangle \\ \times |x_S\rangle\langle x'_S| \otimes |x_A + y_A\rangle\langle x'_A + y'_A| \tag{6.33}$$

where we have used the relation $\exp(-ia\hat{p}_A)|x_A\rangle = |x_A + a_A\rangle$.

The postmeasurement state $\hat{\rho}^S_{\text{out}}(r)$ of the physical system after the measurement outcome r was obtained and the probability density $P^A_{\text{out}}(r)$ of the measurement outcome r are given respectively by

$$\hat{\rho}^S_{\text{out}}(r) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |x_S\rangle \left[\frac{\langle x_S|\hat{\rho}^S_{\text{in}}|y_S\rangle\langle r_A - x_A|\hat{\rho}^A_{\text{in}}|r_A - y_A\rangle}{P^A_{\text{out}}(r)}\right]\langle y_S| \tag{6.34}$$

$$P^A_{\text{out}}(r) = \int_{-\infty}^{\infty} dx \; \langle x_S|\hat{\rho}^S_{\text{in}}|x_S\rangle\langle r_A - x_A|\hat{\rho}^A_{\text{in}}|r_A - x_A\rangle \tag{6.35}$$

Since we have the initial probability density $P^S_{\text{in}}(x) = \langle x_S|\hat{\rho}^S_{\text{in}}|x_S\rangle$ of the physical system, we find from (6.35) that the conditional probability density $P_{SA}(r|x)$ becomes

$$P_{SA}(r|x) = \langle r_A - x_A|\hat{\rho}^A_{\text{in}}|r_A - x_A\rangle \tag{6.36}$$

which gives the relation

$$P^A_{\text{out}}(r) = \int_{-\infty}^{\infty} dx \; P_{SA}(r|x)P^S_{\text{in}}(x) \tag{6.37}$$

Furthermore, the operators $\hat{U}_A(x, y) = \langle x_S|\hat{\mathcal{U}}_{SA}|y_S\rangle$ and $\hat{U}^\dagger_A(x, y) = \langle x_S|\hat{\mathcal{U}}^\dagger_{SA}|y_S\rangle$ of the measurement apparatus satisfy the relation

$$\text{Tr}_A[\hat{U}^\dagger_A(x,x')\hat{E}^A_y(r)\hat{U}_A(x'', x''')\hat{\rho}^A_{\text{in}}] \\ = \delta(x - x')\delta(x'' - x''')\langle r_A - x''_A|\hat{\rho}^A_{\text{in}}|r_A - x_A\rangle \tag{6.38}$$

It is easy to see from this relation that Conditions 6.1 and 6.2 with $f(x; y) = x$ are fulfilled. Therefore Theorems 6.1 and 6.2 are established in the position measurement of the physical system. Furthermore the POVM $\hat{\Pi}_S(r)$ of the physical system is given by

$$\hat{\Pi}_S(r) = \int_{-\infty}^{\infty} dx \; |x_S\rangle P_{SA}(r|x)\langle x_S| \tag{6.39}$$

from which the operational observable $\hat{\chi}_S^{\text{op}}(n)$ of the physical system defined by the position measurement is calculated to be

$$\hat{\chi}_S^{\text{op}}(n) = \int_{-\infty}^{\infty} dx \; (\hat{x}_S + x)^n \langle x_A|\hat{\rho}_{\text{in}}^A|x_A\rangle \tag{6.40}$$

It is obvious that this operational observable commutes with the intrinsic observable $\hat{x}_S$ of the physical system.

Since Theorem 6.2 holds in the position measurement of the physical system, we have the equality $I(Y_{\text{out}}^A; X_{\text{in}}^S) = \Delta H(X_{\text{out}}^S, X_{\text{in}}^S|Y_{\text{out}}^A)$. Here we show this equality by explicit calculation of the entropy decrease $\Delta H(X_{\text{out}}^S, X_{\text{in}}^S|Y_{\text{out}}^A)$ of the physical system. This is an easy task when we use the following expression of the joint probability density $P_{\text{out}}^{SA}(x, y)$ in the compound quantum state $\hat{\rho}_{\text{out}}^{SA}$ after the interaction between the physical system and the measurement apparatus:

$$\begin{aligned} P_{\text{out}}^{SA}(x, y) &= \text{Tr}_{SA}[(|x_S\rangle\langle x_S| \otimes |y_A\rangle\langle y_A|)\hat{\rho}_{\text{out}}^{SA}] \\ &= \langle y_A - x_A|\hat{\rho}_{\text{in}}^A|y_A - x_A\rangle\langle x_S|\hat{\rho}_{\text{in}}^S|x_S\rangle \\ &= P_{SA}(y|x)P_{\text{in}}^S(x) \end{aligned} \tag{6.41}$$

which is equivalent to (6.29) with $\mu = x$, $\nu = y$, and $f(x, y) = x$. Then we can calculate the joint entropy $H(X_{\text{out}}^S, Y_{\text{out}}^A)$ as follows:

$$\begin{aligned} &H(X_{\text{out}}^S, Y_{\text{out}}^A) \\ &= -\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; P_{SA}(y|x)P_{\text{in}}^S(x) \ln[P_{SA}(y|x)P_{\text{in}}^S(x)] \\ &= H(X_{\text{in}}^S) - \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; P_{SA}(y|x)P_{\text{in}}^S(x) \ln P_{SA}(y|x) \\ &= H(X_{\text{in}}^S) + H(Y_{\text{out}}^A) - \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; P_{SA}(y|x)P_{\text{in}}^S(x) \ln\left[\frac{P_{SA}(y|x)}{P_{\text{out}}^A(y)}\right] \\ &= H(X_{\text{in}}^S) + H(Y_{\text{out}}^A) - I(Y_{\text{out}}^A; X_{\text{in}}^S) \end{aligned} \tag{6.42}$$

where we have used (6.22) and (6.37). Thus we have found from (4.2) that the entropy decrease of the physical system is equal to the amount of information extracted from the measurement outcomes, $I(Y^A_{\text{out}}; X^S_{\text{in}}) = \Delta H(X^S_{\text{out}}, X^S_{\text{in}} | Y^A_{\text{out}})$.

Finally let us rewrite the amount of information about the position observable $\hat{x}_S$ obtained from the measurement outcomes in another form. Using (6.22), (6.36), and (6.37), we can calculate $I(Y^A_{\text{out}}; X^S_{\text{in}})$ as follows:

$$\begin{aligned} I(Y^A_{\text{out}}; X^S_{\text{in}}) &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, P_{SA}(x|y) P^S_{\text{in}}(y) \ln\left[\frac{P_{SA}(x|y)}{P^A_{\text{out}}(x)}\right] \\ &= H(Y^A_{\text{out}}) + \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \langle x_A - y_A | \hat{\rho}^A_{\text{in}} | x_A - y_A \rangle P^S_{\text{in}}(y) \\ &\quad \times \ln\langle x_A - y_A | \hat{\rho}^A_{\text{in}} | x_A - y_A \rangle \\ &= H(Y^A_{\text{out}}) + \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \langle x_A | \hat{\rho}^A_{\text{in}} | x_A \rangle P^S_{\text{in}}(y) \ln\langle x_A | \hat{\rho}^A_{\text{in}} | x_A \rangle \\ &= H(Y^A_{\text{out}}) - H(Y^A_{\text{in}}) \end{aligned} \tag{6.43}$$

where $H(Y^A_{\text{in}})$ is the differential entropy of the measurement apparatus in the initial quantum state $\hat{\rho}^A_{\text{in}}$,

$$H(Y^A_{\text{in}}) = -\int_{-\infty}^{\infty} dx\, \langle x_A | \hat{\rho}^A_{\text{in}} | x_A \rangle \ln\langle x_A | \hat{\rho}^A_{\text{in}} | x_A \rangle \tag{6.44}$$

This result indicates that the amount of information about the position observable $\hat{x}_S$ obtained from the measurement outcomes is equal to the entropy increase of the measurement apparatus in the quantum measurement process.

We have investigated the quantum measurement process of the canonical position observable $\hat{x}_S$ of the physical system. The same results are also obtained for the quantum measurement process of the canonical momentum observable of the physical system. In the momentum measurement, the intrinsic observable of the physical system and the pointer observable of the measurement apparatus are canonical momentum operators,

$$\hat{\chi}_S = \hat{p}_S, \qquad \mathcal{Y}_A = \hat{p}_A \tag{6.45}$$

The unitary operator $\mathcal{U}_{SA}$ that describes the state change caused by the interaction between the physical system and the measurement apparatus is given by

$$\mathcal{U}_{SA} = \exp(-i\hat{p}_S\hat{x}_A) \tag{6.46}$$

In this case, since the quantum measurement process of the momentum observable satisfies Conditions 6.1 and 6.2, we obtain Theorems 6.1 and 6.2. Therefore the amount of information about the momentum observable $\hat{p}_S$ of the physical system extracted from the measurement outcomes is equal to the entropy decrease of the physical system, $I(Y^A_{\text{out}}; X^S_{\text{in}}) = \Delta H(X^S_{\text{out}}, X^S_{\text{in}}|Y^A_{\text{out}})$, and to the entropy increase of the measurement apparatus in the quantum measurement process, $I(Y^A_{\text{out}}; X^S_{\text{in}}) = H(Y^A_{\text{out}}) - H(Y^A_{\text{in}})$.

7. CONTINUOUS MEASUREMENTS

We have considered the amount of information extracted from the measurement outcomes and the entropy change of the physical system in the quantum measurement processes of discrete and continuous observables. The quantum measurement processes in Sections 5 and 6 have used the projection operators to obtain the results of the quantum measurement process, though the general results obtained in Sections 3, 4, and 6 are valid for using any POVM to obtain the measurement outcomes. Therefore we will consider a quantum measurement process in which the readout of the measurement outcomes cannot be described by any projection operator, or equivalently in which the pointer observable of the measurement apparatus cannot be defined. In this section we use the photon counting measurement (continuous quantum measurement of photon number), which obeys the quantum Markov process (Srinivas and Davies, 1981; Srinivas, 1996, Chmara, 1987; Ban, 1997b), to obtain the photon number of the measurement apparatus. As an example, we consider the degenerate four-wave mixing process (Ban, 1996b) with the photon counting measurement, which corresponds to the continuous quantum nondemolition measurement of the photon number of the physical system (Braginsky and Khalili, 1992).

7.1. Photon Counting Measurement

The photon counting measurement, which obeys the quantum Markov process, consists of two basic processes, a one-count process and a no-count process (Srinivas and Davies, 1981; Srinivas, 1996, Chmara, 1987; Ban, 1997b). Let $\hat{\rho}^{SA}_{\text{out}}$ be the compound quantum state of the physical system and the measurement apparatus after the interaction between them. Then we perform the photon counting measurement on the measurement apparatus to obtain the value of the photon number operator $\hat{n}_A = \hat{a}^\dagger_A \hat{a}_A$. The one-count process represents the state change which occurs when the photodetector registers one photon of the measurement apparatus. This process is described by the superoperator $\mathfrak{I}_A$ of the measurement apparatus. The state change caused by the one-count process and the probability that the one-count process occurs in an infinitesimal time interval dt are given respectively by

$$\hat{\rho}^{SA}_{\text{out}} \to \frac{\mathcal{T}_A \hat{\rho}^{SA}_{\text{out}}}{\text{Tr}_{SA}[\mathcal{T}_A \hat{\rho}^{SA}_{\text{out}}]}, \qquad P(1;\, dt) = \text{Tr}_{SA}[\mathcal{T}_A \hat{\rho}^{SA}_{\text{out}}]\, dt \tag{7.1}$$

The no-count process represents the time evolution of the quantum state during which the photodetector does not register any photon of the measurement apparatus. The time evolution of the quantum state in the no-count process and the probability that the no-count process continues during time t are given respectively by

$$\hat{\rho}^{SA}_{\text{out}} \to \frac{\exp(t\mathcal{L}_A)\hat{\rho}^{SA}_{\text{out}}}{\text{Tr}_{SA}[\exp(t\mathcal{L}_A)\hat{\rho}^{SA}_{\text{out}}]}, \qquad P(0;\, t) = \text{Tr}_{SA}[\exp(t\mathcal{L}_A)\hat{\rho}^{SA}_{\text{out}}] \tag{7.2}$$

where the generator $\mathcal{L}_A$ is the superoperator of the measurement apparatus. In this equation, we have ignored the time evolution of the system that is independent of the photon counting measurement, for the sake of simplicity.

It is assumed in the photon counting process that the photodetector cannot register more than one photon in an infinitesimal time interval dt. In this case, the normalization condition of the photon counting probability is given by $P(0;\, dt) + P(1;\, dt) = 1$, which yields the relation between the superoperators $\mathcal{T}_A$ and $\mathcal{L}_A$,

$$\text{Tr}_{SA}[(\mathcal{T}_A + \mathcal{L}_A)\hat{\rho}^{SA}_{\text{out}}] = 0 \tag{7.3}$$

This relation is used to determine the superoperators $\mathcal{T}_A$ and $\mathcal{L}_A$. Using the superoperators $\mathcal{T}_A$ and $\mathcal{L}_A$, we can describe the m-count process that the photodetector registers m photons of the measurement apparatus during time t. The superoperator $\mathcal{N}^A_m(t)$ of the m-count process is given by

$$\begin{aligned}\mathcal{N}^A_m(t) = &\int_0^t dt_m \int_0^{t_m} dt_{m-1} \cdots \int_0^{t_2} dt_1 \\ &\times \hat{S}_A(t - t_m)\, \mathcal{T}_A \hat{S}_A(t_m - t_{m-1})\, \mathcal{T}_A \cdots \hat{S}_A(t_2 - t_1)\, \mathcal{T}_A \hat{S}_A(t_1)\end{aligned} \tag{7.4}$$

where we have defined the superoperator $\hat{S}_A(t) = \exp(t\mathcal{L}_A)$. In this equation the integrand represents the process that the photodetector registers one photon at each of the times $t_1, t_2, \ldots, t_m$ and does not register any photon in the rest of the time interval $(0, t)$. The compound quantum state $\hat{\rho}^{SA}_{\text{out}}(m)$ of the physical system and the measurement apparatus after the m-count process and the probability $P^A_{\text{out}}(m)$ that the m-count process occurs are given respectively by

$$\hat{\rho}^{SA}_{\text{out}}(m) = \frac{\mathcal{N}^A_m(t)\, \hat{\rho}^{SA}_{\text{out}}}{\text{Tr}_{SA}[\mathcal{N}^A_m(t)\hat{\rho}^{SA}_{\text{out}}]}, \qquad P^A_{\text{out}}(m) = \text{Tr}_{SA}[\mathcal{N}^A_m(t)\, \hat{\rho}^{SA}_{\text{out}}] \tag{7.5}$$

where the probability $P^A_{\rm out}(m)$ is normalized as

$$\sum_{m=0}^{\infty} P^A_{\rm out}(m) = {\rm Tr}_{SA}[e^{t(\mathcal{L}_A + \mathcal{T}_A)}) \hat{\rho}^{SA}_{\rm out}] = 1$$

which is ensured by (7.3).

To investigate the photon counting measurement, we have to determine the superoperators $\mathcal{T}_A$ and $\mathcal{L}_A$ explicitly. For this purpose, we assume here that when the photodetector registers one photon of the measurement apparatus, the photon disappears from the measurement apparatus (Srinivas and Davies, 1981; Srinivas, 1996). Under this assumption, the superoperator $\mathcal{T}$ of the one-count process is given by

$$\mathcal{T}_A\hat{\rho}^{SA}_{\rm out} = \lambda\hat{a}_A\hat{\rho}^{SA}_{\rm out}\hat{a}^\dagger_A \tag{7.6}$$

where the parameter λ represents the strength of the interaction between the photon of the measurement apparatus and the photodetector. Of course, we can use a different superoperator to describe the one-count process such as $\mathcal{T}_A\hat{\rho}^{SA}_{\rm out} = \lambda\hat{a}^\dagger_A\hat{\rho}^{SA}_{\rm out}\hat{a}_A$ or $\mathcal{T}_A\hat{\rho}^{SA}_{\rm out} = \lambda\hat{a}^\dagger_A\hat{a}_A\hat{\rho}^{SA}_{\rm out}\hat{a}^\dagger_A\hat{a}_A$. The former represents the photon counting measurement with Mandel's quantum counter (Mandel, 1966; Ueda and Kitagawa, 1992) and the latter represents the continuous quantum nondemolition measurement of the photon number (Ueda *et al.*, 1992). Even if we use these superoperators, we can obtain the results in the same way. Using relation (7.3) and the fact that $\hat{S}_A(t)\hat{\rho}^{SA}_{\rm out}$ should be a Hermitian operator, we obtain the superoperator $\mathcal{L}_A$ from (7.6),

$$\mathcal{L}_A\hat{\rho}^{SA}_{\rm out} = -\frac{1}{2}\lambda(\hat{a}^\dagger_A\hat{a}_A\hat{\rho}^{SA}_{\rm out} + \hat{\rho}^{SA}_{\rm out}\hat{a}^\dagger_A\hat{a}_A) \tag{7.7}$$

Thus the compound quantum state $\hat{\rho}^{SA}_{\rm out}(m)$ after the m-count process and the photon-counting probability $P^A_{\rm out}(m)$ become

$$\hat{\rho}^{SA}_{\rm out}(m;\gamma) = \frac{\exp(-\frac{1}{2}\lambda t\hat{a}^\dagger_A\hat{a}_A)\hat{a}^m_A\hat{\rho}^{SA}_{\rm out}\hat{a}^{\dagger m}_A\exp(-\frac{1}{2}\lambda t\hat{a}^\dagger_A\hat{a}_A)}{{\rm Tr}_{SA}[\hat{a}^{\dagger m}_A\exp(-\lambda t\hat{a}^\dagger_A\hat{a}_A)\hat{a}^m_A\hat{\rho}^{SA}_{\rm out}]} \tag{7.8}$$

$$\begin{aligned} P^A_{\rm out}(m;\gamma) &= \frac{1}{m!}\gamma^m{\rm Tr}_{SA}[\hat{a}^{\dagger m}_A\exp(-\gamma\hat{a}^\dagger_A\hat{a}_A)\hat{a}^m_A\hat{\rho}^{SA}_{\rm out}] \\ &= \sum_{n=m}^{\infty}\frac{n!}{m!(n-m)!}\gamma^m(1-\lambda)^{n-m}\langle n_A|{\rm Tr}_S[\hat{\rho}^{SA}_{\rm out}]|n_A\rangle \end{aligned} \tag{7.9}$$

where the parameter γ is called the effective quantum efficiency of the photodetector (Srinivas and Davies, 1981; Srinivas, 1996),

$$\gamma = 1 - \exp(-\lambda t) \tag{7.10}$$

In equations (7.8) and (7.9), we have written the quantum state and the probability as $\hat{\rho}^{SA}_{\text{out}}(m;\,\gamma)$ and $P^{A}_{\text{out}}(m;\,\gamma)$ to emphasize that they are obtained by the photon counting process with the photodetector of effective quantum efficiency γ.

Let us now introduce an operator $\hat{E}^{A}_{\mathcal{Y}}(m;\,\gamma)$ of the measurement apparatus,

$$\hat{E}^{A}_{\mathcal{Y}}(m\gamma) = \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!}\,\gamma^{m}(1-\gamma)^{n-m}|n_A\rangle\langle n_A|$$

$$= \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!}\,\gamma^{m}(1-\gamma)^{n-m}\,\hat{E}^{A}_{\mathcal{Y}}(n) \qquad (7.11)$$

where $\hat{E}^{A}_{\mathcal{Y}}(n) = |n_A\rangle\langle n_A|$. This operator satisfies the relations

$$\hat{E}^{A}_{\mathcal{Y}}(m;\,\gamma) \geq 0, \qquad \sum_{m=0}^{\infty} \hat{E}^{A}_{\mathcal{Y}}(m;\,\gamma) = \hat{I}_A \qquad (7.12)$$

Thus the operator $\hat{E}^{A}_{\mathcal{Y}}(m;\,\gamma)$ is nothing but the POVM of the measurement apparatus. Using the POVM $\hat{E}^{A}_{\mathcal{Y}}(m;\,\gamma)$, we can express the probability $P^{A}_{\text{out}}(m;\,\gamma)$ of the measurement outcome m in the photon counting process,

$$P^{A}_{\text{out}}(m;\,\gamma) = \text{Tr}_{SA}[(\hat{I}_S \otimes \hat{E}^{A}_{\mathcal{Y}}(m;\,\gamma))\hat{\rho}^{SA}_{\text{out}}] \qquad (7.13)$$

The postmeasurement state $\hat{\rho}^{S}_{\text{out}}(m;\,\gamma)$ of the physical system after the measurement outcome m was obtained is derived from (7.8),

$$\hat{\rho}^{S}_{\text{out}}(m;\,\gamma) = \text{Tr}_A[\hat{\rho}^{SA}_{\text{out}}(m;\,\gamma)] = \frac{\text{Tr}_A[(\hat{I}_S \otimes \hat{E}^{A}_{\mathcal{Y}}(m;\,\gamma))\hat{\rho}^{SA}_{\text{out}}]}{\text{Tr}_{SA}[(\hat{I}_S \otimes \hat{E}^{A}_{\mathcal{Y}}(m;\,\gamma))\hat{\rho}^{SA}_{\text{out}}]} \qquad (7.14)$$

It is important to note that equations (7.13) and (7.14) are the same as Eqs. (2.7) and (2.6). Therefore the general results obtained in Sections 3 and 4 hold when we perform the photon counting measurement to obtain the photon number of the measurement apparatus. The POVM of the measurement apparatus for the photon counting measurement is given by (7.11) and it is not a projection operator. It should be noted that if the effective quantum efficiency γ is unity, the POVM $\hat{E}^{A}_{\mathcal{Y}}(m;\,\gamma)$ becomes the projection operator onto the eigenspace of the photon number operator $\hat{a}^{\dagger}_A\hat{a}_A$, that is, $\lim_{\gamma\to 1}\hat{E}^{A}_{\mathcal{Y}}(m;\,\gamma) = \hat{E}^{A}_{\mathcal{Y}}(m) = |m_A\rangle\langle m_A|$.

7.2. Degenerate Four-Wave Mixing Process

We now consider the degenerate four-wave mixing process with the photon counting measurement to examine the results obtained above. Suppose that we obtain information about the photon number of the physical system by means of this quantum measurement process. In this case we have the intrinsic observable $\hat{\chi}_S = \hat{a}^{\dagger}_S\hat{a}_S$, the spectral set $\mathcal{M} = \{0, 1, 2, \ldots, \infty\}$, and the projection operator

$\hat{E}^S_\chi(n) = |n_S\rangle\langle n_S|$. The quantum measurement process which is characterized by the triplet $\mathbb{M} = \langle\hat{\rho}^A_{\text{in}}, \hat{E}^A_{\mathcal{Y}}(\nu), \mathcal{U}_{SA}\rangle$ is set up in the following way.

1. The measurement apparatus is prepared in the vacuum state $\hat{\rho}^A_{\text{in}} = |0_A\rangle\langle 0_A|$ before the interaction with the physical system.
2. The photon number of the measurement apparatus is obtained by means of the photon counting measurement, which obeys the quantum Markov process. The readout of the photon number is described by the POVM $\hat{E}^A_{\mathcal{Y}}(m; \gamma)$ [see (7.11)], and we have $\mathcal{N} = \{0, 1, 2, \ldots, \infty\}$.
3. The unitary operator $\mathcal{U}_{SA}$ that describes the state change caused by the interaction between the physical system and the measurement apparatus is given by

$$\mathcal{U}_{SA} = \exp[-ig^{1/2}\hat{a}^\dagger_S\hat{a}_S\,(\hat{a}_A + \hat{a}^\dagger_A)] \tag{7.15}$$

where the parameter g represents the coupling constant of the degenerate four-wave mixing process (Milburn and Walls, 1984; Ban, 1996b).

In the degenerate four-wave mixing process, the compound quantum state of the physical system and the measurement apparatus after the interaction becomes

$$\hat{\rho}^{SA}_{\text{out}} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-i)^m i^n}{\sqrt{m!n!}}\exp(-\tfrac{1}{2}g\hat{n}_S^2)(g^{1/2}\hat{n}_S)^m\,\hat{\rho}^S_{\text{in}}(g^{1/2}\hat{n}_S)^n$$
$$\times\exp(-\tfrac{1}{2}g\hat{n}_S^2)\otimes|m_A\rangle\langle n_A| \tag{7.16}$$

where $\hat{n}_S = \hat{a}^\dagger_S\hat{a}_S$ is the photon number operator of the physical system.

Substituting (7.11) and (7.16) into (7.13) and (7.14), we obtain the postmeasurement state $\hat{\rho}^S_{\text{out}}(m; \gamma)$ of the physical system after we obtained the measurement outcome m and the probability $P^A_{\text{out}}(m; \gamma)$ of the measurement outcome m,

$$\hat{\rho}^S_{\text{out}}(m) = \frac{\langle\exp(-\frac{1}{2}g\hat{n}_S^2)\hat{n}_S^n\hat{\rho}^S_{\text{in}}\hat{n}_S^n\exp(-\frac{1}{2}g\hat{n}_S^2)\rangle_{(m;\gamma)}}{\langle\text{Tr}_S[\exp(-g\hat{n}_S^2)\hat{n}_S^{2n}\hat{\rho}^S_{\text{in}}]\rangle_{(m;\gamma)}} \tag{7.17}$$

$$P^A_{\text{out}}(m; \gamma) = \left\langle\frac{1}{n!}g^n\text{Tr}_S[\exp(-g\hat{n}_S^2)\hat{n}_S^{2n}\,\hat{\rho}^S_{\text{in}}]\right\rangle_{(m;\gamma)} \tag{7.18}$$

where $\langle F(n)\rangle_{(m;\gamma)}$ represents the average value of $F(n)$ by means of the binomial distribution obtained in the photon counting measurement,

$$\langle F(n)\rangle_{(m;\gamma)} = \sum_{n=m}^{\infty}\frac{n!}{m!(n-m)!}\gamma^m(1-\gamma)^{n-m}F(n) \tag{7.19}$$

The operators $\hat{U}_A(n, n')$ and $\hat{U}_A^\dagger(n, n')$ of the measurement apparatus, which are given by (3.24) and (3.25), satisfy the relation

$$\mathrm{Tr}_A[\hat{U}_A^\dagger(n_1, n_2)\, \hat{E}_{\mathcal{Y}}^A(m; \gamma)\hat{U}_A(n_3, n_4)\hat{\rho}_{\mathrm{in}}^A]$$

$$= \delta_{n_1, n_2}\delta_{n_3, n_4} \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!}\, \gamma^m(1-\gamma)^{n-m} \sqrt{H(n, n_1)\, H(n, n_3)} \quad (7.20)$$

where $H(m, n)$ is given by

$$H(m, n) = \frac{1}{m!}(gn^2)^m \exp(-gn^2) \quad (7.21)$$

It is easy to see from equation (7.20) that the degenerate four-wave mixing process with the photon counting measurement satisfies Conditions 3.1 and 4.1–4.3 with $f(n; m) = n$. Therefore it is found from Theorem 4.1 that the amount of information about the photon number of the physical system extracted from the measurement outcomes is equal to the entropy decrease of the physical system in the quantum measurement process, that is, $I(Y_{\mathrm{out}}^A; X_{\mathrm{in}}^S) = \Delta H(X_{\mathrm{out}}^S, X_{\mathrm{in}}^S | Y_{\mathrm{out}}^A)$. If the premeasurement state $\hat{\rho}_{\mathrm{in}}^S$ of the physical system is diagonal with respect to the photon-number eigenstate $|n_S\rangle$, the postmeasurement state $\hat{\rho}_{\mathrm{out}}^S(m; \gamma)$ becomes diagonal. In this case the decrease of the Shannon entropy is equal to that of the von Neumann entropy, $\Delta H(X_{\mathrm{out}}^S, X_{\mathrm{in}}^S | Y_{\mathrm{out}}^A) = \Delta S(X_{\mathrm{out}}^S, X_{\mathrm{in}}^S | Y_{\mathrm{out}}^A)$ (see Theorem 4.2).

The POVM $\hat{\Pi}_S(m; \gamma)$ of the physical system in the degenerate four-wave mixing process with the photon counting measurement is obtained from (3.29) and (7.20),

$$\hat{\Pi}_S(m; \gamma) = \sum_{n=0}^{\infty} \sum_{k=m}^{\infty} \frac{k!}{m!(k-m)!}\, \gamma^m(1-\gamma)^{k-m}\, H(k, n)\, |n_S\rangle\langle n_S| \quad (7.22)$$

The conditional probability $P_{SA}(m|n; \gamma)$ that the measurement outcome is m when the photon number observable takes the value n in the premeasurement state of the physical system becomes

$$\begin{aligned} P_{SA}(m|n; \gamma) &= \langle n_S|\hat{\Pi}_S(m; \gamma)|n_S\rangle \\ &= \sum_{k=m}^{\infty} \frac{k!}{m!(k-m)!}\, \gamma^m(1-\gamma)^{k-m}\, H(k, n) \\ &= \frac{1}{m!}(\gamma g n^2)^m \exp(-\gamma g n^2) \end{aligned} \quad (7.23)$$

which satisfies the relation

$$P_{\mathrm{out}}^A(m; \gamma) = \sum_{n=0}^{\infty} P_{SA}(m|n; \gamma)\, P_{\mathrm{in}}^S(n) \quad (7.24)$$

where $P_{\mathrm{in}}^{S}(n) = \langle n_S|\hat{\rho}_{\mathrm{in}}^{S}|n_S\rangle$ and $P_{\mathrm{out}}^{A}(m;\gamma)$ is given by equation (7.18). Furthermore, using the POVM $\hat{\Pi}_S(m;\gamma)$, we obtain the operational observable $\mathcal{N}_S^{\mathrm{op}}(n;\gamma)$ of the physical system defined by the quantum measurement process,

$$\mathcal{N}_S^{\mathrm{op}}(n;\gamma) = \sum_{m=0}^{\infty} m^n \hat{\Pi}_S(m;\gamma) = \frac{\partial^n}{\partial \xi^n}\mathcal{G}_S(\xi;\gamma)\bigg|_{\xi=0} \tag{7.25}$$

with

$$\mathcal{G}_S(\xi;\gamma) = \exp[g\gamma(e^{\xi}-1)\hat{n}_S^2] \tag{7.26}$$

In particular we obtain for $n = 1$ and $n = 2$

$$\mathcal{N}_S^{\mathrm{op}}(1;\gamma) = g\gamma\hat{n}_S^2, \qquad \mathcal{N}_S^{\mathrm{op}}(2;\gamma) = g\gamma\hat{n}_S^2\,(g\gamma\hat{n}_S^2 + 1) \tag{7.27}$$

These results indicate that the operational observable of the physical system decreases by a factor γ times that obtained in the ideal photon number measurement, which is described by the projection operator $\hat{E}_{\mathcal{Y}}^{A}(n) = |n_A\rangle\langle n_A|$.

Finally we explicitly calculate the entropy decrease of the physical system to check the equality $I(Y_{\mathrm{out}}^{A}; X_{\mathrm{in}}^{S}) = \Delta H(X_{\mathrm{out}}^{S}, X_{\mathrm{in}}^{S}|Y_{\mathrm{out}}^{A})$. When we use the relation

$$\langle n_S|\hat{\rho}_{\mathrm{out}}^{S}(m;\gamma)|n_S\rangle = \frac{P_{SA}(m|n;\gamma)\,P_{\mathrm{in}}^{S}(n)}{P_{\mathrm{out}}^{A}(m;\gamma)} \tag{7.28}$$

we can calculate the entropy decrease $\Delta H_\gamma(X_{\mathrm{out}}^{S}, X_{\mathrm{in}}^{S}|Y_{\mathrm{out}}^{A})$ as follows:

$$\begin{aligned}
&\Delta H(X_{\mathrm{out}}^{S}, X_{\mathrm{in}}^{S}|Y_{\mathrm{out}}^{A})\\
&\quad = H(X_{\mathrm{in}}^{S}) - H(X_{\mathrm{out}}^{S}|Y_{\mathrm{out}}^{A})\\
&\quad = -\sum_{n=0}^{\infty} P_{\mathrm{in}}^{S}(n)\ln P^{S}{}_{\mathrm{in}}(n) + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_{\mathrm{out}}^{A}(m;\gamma)\\
&\qquad \times \langle n_S|\hat{\rho}_{\mathrm{out}}^{S}(m;\gamma)|n_S\rangle \ln\langle n_S|\hat{\rho}_{\mathrm{out}}^{S}(m;\gamma)|n_S\rangle\\
&\quad = -\sum_{n=0}^{\infty} P_{\mathrm{in}}^{S}(n)\ln P_{\mathrm{in}}^{S}(n) + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_{SA}(m|n;\gamma)\,P_{\mathrm{in}}^{S}(n)\\
&\qquad \times \ln\left[\frac{P_{SA}(m|n;\gamma)P_{\mathrm{in}}^{S}(n)}{P_{\mathrm{out}}^{A}(m;\gamma)}\right]\\
&\quad = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_{SA}(m|n;\gamma)P_{\mathrm{in}}^{S}(n)\ln\left[\frac{P_{SA}(m|n;\gamma)}{P_{\mathrm{out}}^{A}(m;\gamma)}\right]\\
&\quad = I(Y_{\mathrm{out}}^{A}; X_{\mathrm{in}}^{S})
\end{aligned} \tag{7.29}$$

This result explicitly shows the validity of Theorem 4.1.

8. SUMMARY

In this paper we have investigated the amount of information about the intrinsic observable of a physical system that can be extracted from the results of the quantum measurement process and we have also considered the decrease of the Shannon entropy of a measured physical system caused by the quantum measurement process. When the operational observable of the physical system defined by the quantum measurement process commutes with the intrinsic observable of the physical system, the amount of information about the intrinsic observable can be expressed by the mutual information between the physical system and the measurement apparatus. If the quantum measurement process which is characterized by the triplet $\mathbb{M} = \langle \hat{\rho}^A_{\text{in}}, \hat{E}^A_{\mathcal{Y}}(\nu), \hat{\mathcal{U}}_{SA} \rangle$ satisfies Conditions 4.1–4.3 or Conditions 6.1 and 6.2, the entropy decrease of the physical system caused by the quantum measurement process becomes equal to the information gain. Furthermore, it has been shown in the quantum measurement processes of discrete observables that if the statistical operator of the postmeasurement state (the premeasurement state) of the physical system commutes with the intrinsic observable of the physical system, the decrease of the Shannon entropy is no less (no greater) than that of the von Neumann entropy in the quantum measurement process. The main results obtained in this paper are summarized in Theorems 3.1, 4.1, 4.2, 6.1, and 6.2.

We have considered several examples of quantum measurement processes to examine the general results. The normal unitary process which satisfies the probability reproducibility condition and the SU(2) and SU(1, 1) processes with the photon number measurement in quantum optical systems have been considered as examples that satisfy Conditions 4.1–4.3. Furthermore, the position and momentum measurements of a physical system have been considered to show that the general results are still valid for quantum measurement processes of continuous observables. In these quantum measurement processes, the readout of the measurement outcomes is performed by measuring the pointer observable of the measurement apparatus, which has a discrete or continuous spectrum. The general results obtained in this paper, however, are established in quantum measurement processes where the pointer observable of the measurement apparatus cannot be defined. To show this explicitly, we have considered the quantum measurement process in which the readout of the measurement outcome is performed by the photon counting measurement (the continuous measurement of photon number), which obeys the quantum Markov process. As an example, we have investigated the degenerate four-wave mixing process with the photon counting measurement, which is equivalent to the continuous quantum nondemolition measurement of the photon number of a physical system. Therefore we have found that the general results obtained in this paper are valid for many kinds of quantum measurement processes.

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